Some inequalities related to transience and recurrence of Markov processes and their applications

Daehong Kim* and Yoichi Oshima†

Abstract

For an irreducible symmetric Markov process on a, not necessarily compact, state space associated with a symmetric Dirichlet form, we give Poincaré type inequalities. As an application of the inequalities, we consider a time inhomogeneous diffusion process obtained by a time dependent drift transformation from a diffusion process and give general conditions for the transience or recurrence of some sets. As a particular case, the diffusion process appearing in the theory of simmulated annealing is considered.

Keywords : Dirichlet forms, Poincaré type inequality, Recurrent process, Simulated annealing, Time inhomogeneous diffusion processes

Mathematics Subject Classification 2000 : 60J60, 60J45, 60H30, 31C25

1 Introduction

Let X be a locally compact separable metric space and m a positive Radon measure on X with full support. Consider an irreducible regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X;m)$ and its associated m-symmetric Markov process $\mathbf{M} = (X_t, P_x)$ on X. $\mathbf{M}$ is called transient if there exists a strictly positive function $g \in L^1(X;m)$ such that $Rg(x) = E_x\left(\int_0^\infty g(X_t)dt\right) < \infty$ for a.e. $x \in X$. $\mathbf{M}$ is called recurrent if it is not transient or, equivalently, if $P_x(\sigma_F < \infty) = 1$ q.e. $x \in X$ for any non-exceptional set F in X, where σ_F is the hitting time of F.

Using the Dirichlet form, transience of $\mathbf{M}$ is characterized as follows : $\mathbf{M}$ is transient if and only if there exists a strictly positive function $g \in L^1(X;m)$ and a constant $k_1(g)$ such that

$$\int_X |u(x)|g(x)dm(x) \le k_1(g)\mathcal{E}(u,u)^{1/2}, \quad u \in \mathcal{F} \tag{1}$$

*This research was partially supported by Grant-in-Aid for Scientific Research No.17740062

†This research was partially supported by Grant-in-Aid for Scientific Research No.18540131

([4]). As an L^2-version of (1), the following result also holds (see [3],[12]): For any non-negative bounded m-integrable function g such that $\|Rg\|_\infty < \infty$,

$$\int_X u^2(x)g(x)dm(x) \le 2\|Rg\|_\infty \mathcal{E}(u,u), \quad u \in \mathcal{F}. \tag{2}$$

In particular, if $\|R1\|_\infty < \infty$, then (2) holds for $g = 1$ without the factor 2 in the righthand side. On the other hand, if $\mathbf{M}$ is Harris recurrent, it is known that there exists a strictly positive function $g \in L^1(X;m)$, a non-null set C of X and a constant $k_2(g)$ such that

$$\int_X |u(x) - \langle \nu_C, u\rangle| g(x)dm(x) \le k_2(g)\mathcal{E}(u,u)^{1/2}, \quad u \in \mathcal{F}, \tag{3}$$

where $\nu_C(\cdot) = m(\cdot)/m(C)$ and $\langle \nu_C, u\rangle = \int_X u(x)\, d\nu_C(x)$.

For a given set $F \subset X$, we say that F is a recurrent set of $\mathbf{M}$ if $P_x(\sigma_F < \infty) = 1$ for a.e. $x \in X$. In this case, $\lim_{T\to\infty} P_x(\sigma_F \circ \theta_T < \infty) = 1$ for a.e. $x \in X$. If this limit vanishes, then we call F a transient set of $\mathbf{M}$.

In this paper, we consider some inequalities of Poincaré type related to recurrent Markov processes and apply them to certain time inhomogeneous diffusion process to give general criteria for the transience and recurrence of some sets.

As a particular case, if we assume that $m(X) < \infty$ and there exists a spectral gap $\lambda_1 > 0$ relative to $(\mathcal{E}, \mathcal{F})$ defined by

$$\lambda_1 = \inf_{u\in\mathcal{F}} \frac{\mathcal{E}(u,u)}{\|u - \langle m, u\rangle\|_2^2},$$

then the 1-resolvent R_1 of $\mathbf{M}$ satisfies

$$\|R_1 f - \langle m, f\rangle\|_2 \le \frac{1}{1+\lambda}\|f - \langle m, f\rangle\|_2, \quad f \in L^2(X;m). \tag{4}$$

for any λ such that $0 < \lambda \le \lambda_1$, where $\|\cdot\|_p$ denotes the $L^p(X;m)$-norm. Note that the constant $1/(1+\lambda)$ of the righthand side of (4) is less than 1.

In §2, instead of the existence of a strictly positive lower bound of the spectral gap, we start from the assumption that

$$\sup_{x\in X} \|R_1(x,\cdot) - m(\cdot)\| \le 2\gamma \tag{5}$$

for some $\gamma \in (0,1)$, where $\|\nu\|$ denotes the total variation of the signed measure ν defined by $\|\nu\| = \nu(B^+) - \nu(B^-)$ in terms of the Hahn decomposition $X = B^+ \cup B^-$ relative to ν. In this case, it is easy to see that

$$\|R_1 f - \langle m, f\rangle\|_2 \le 2\gamma\|f - \langle m, f\rangle\|_2, \quad f \in L^2(X;m). \tag{6}$$

But the constant 2γ in the inequality (6) is not optimal in the sense of L^2-setting because it can be greater than one. In Lemma 2.1, we shall show that $(1-\gamma)/\gamma$ is a lower bound of the spectral gap, that is,

$$\int_X (u(x) - \langle m, u\rangle)^2 \, dm(x) \leq \frac{\gamma}{1-\gamma}\mathcal{E}(u,u), \quad u \in \mathcal{F} \tag{7}$$

and as a result the factor 2 in the constant of (6) can be removed. Using this lemma, we shall also give an L^2-version of (3) for general Harris recurrent Markov processes (Theorem 2.1).

There are many interesting features concerning the transience or recurrence of some sets in the time inhomogeneous case because a set can be transient or recurrent depending on the fluctuation of the generator relative to the time parameter, unlike the time homogeneous case.

In §3, we consider the time inhomogeneous diffusion process $\mathbf{M}^\rho = (X_t, P^\rho_{(s,x)})$ associated with the family of energy forms $(\mathcal{E}^{(t)}, \mathcal{F} \cap L^2(X;\mu_t))$ on $L^2(X;\mu_t)$ defined by

$$\mathcal{E}^{(t)}(\varphi,\psi) = \frac{1}{2}\int_X \rho^2(t,x)\, d\mu_{\langle\varphi,\psi\rangle}(x) \tag{8}$$

with a strictly positive time dependent weight function $\rho(t,\cdot) \in \mathcal{F}$, where $d\mu_t(x) = \rho^2(t,x)dm(x)$. As a main result, we give some general criteria on ρ for the transience or recurrence of some sets relative to $\mathbf{M}^\rho$ by applying the inequalities (2) and (7). As an example, we apply our criteria to a Brownian motion $\mathbf{B}$ on a compact connnected Riemannian manifold X and a weight function $\rho(t,x)$ given by

$$\rho(t,x) = \exp\left(-(U(x)/c)\log\sqrt{1+t}\right), \quad c > 0. \tag{9}$$

Indeed, more profound properties of the diffusion $\mathbf{B}^\rho$ can be found in the theory of simulated annealing ([5],[6]).

2 Some inequalities related to transience and recurrence

As is stated in §1, some characterizations of transience and recurrence of symmetric Dirichlet forms $(\mathcal{E},\mathcal{F})$ on $L^2(X;m)$ are given in Fukushima et.al. ([4]). The transience of $(\mathcal{E},\mathcal{F})$ is characterized by (1). Furthermore, in this case, an L^2-version (2) holds.

The purpose of this section is, after getting the inequality (7) for the Markov processes satisfying the inequality (5), to show an L^2-version of (3) for general Harris recurrent Markov processes. To show the inequality (7), we make the following assumptions on $\mathbf{M}$.

(A) **M** is recurrent and there exists $\gamma < 1$ such that (5) holds.

In this case, for any $n \geq 1$,

$$\|R_1^n(x, \cdot) - m(\cdot)\| \leq 2\gamma^n, \quad x \in X \tag{10}$$

([11]). Note that (10) implies that m is a probability measure. The condition **(A)** is satisfied if X is compact and R_1 is strong Feller, or more generally, if the density of the absolutely continuous part of $R_1(x, \cdot)$ relative to m is bounded from below by a positive constant ([11]).

Let $X = B(x)^+ \cup B(x)^-$ be a Hahn decomposition relative to the signed measure $R(x, \cdot) - m(\cdot)$. Then

$$\begin{aligned}\sup_{\|f\|_\infty \leq 1} \|(R_1 - m)f\|_\infty &= \sup_{x \in X} \Big((R_1 - m)I_{B(x)^+} - (R_1 - m)I_{B(x)^-}\Big) \\ &= \sup_{x \in X} \|R_1(x, \cdot) - m(\cdot)\| \leq 2\gamma.\end{aligned}$$

Hence, using the same symbol $\|\cdot\|_p$ to represent the operator norm of $R_1 - m$ in $L^p(X; m)$, we have $\|R_1 - m\|_\infty \leq 2\gamma$. For any $f \in L^1(X; m)$, put $\bar{f} = f - \langle m, f \rangle$ and $B_f = \{x : R_1\bar{f}(x) \geq 0\}$. Then, by the symmetry of R_1, we see

$$\begin{aligned}&\|(R_1 - m)\bar{f}\|_1 \\ &\quad = \int_{B_f} R_1\bar{f}(x)dm(x) - \int_{X \setminus B_f} R_1\bar{f}(x)dm(x) \\ &\quad = \int_X \bar{f}(y)\left\{(R_1(y, B_f) - m(B_f)) - (R_1(y, X \setminus B_f) - m(X \setminus B_f))\right\} dm(y) \\ &\quad \leq \sup_{y \in X} \|R_1(y, \cdot) - m(\cdot)\| \cdot \|\bar{f}\|_1 \\ &\quad \leq 2\gamma \|\bar{f}\|_1\end{aligned}$$

and thus $\|R_1 - m\|_1 \leq 2\gamma$. Denote the total variation measure $|R_1(x, \cdot) - m(\cdot)|$ by

$$\begin{aligned}&|R_1(x, \cdot) - m(\cdot)|(A) \\ &\quad = \Big(R_1(x, A \cap B(x)^+) - m(A \cap B(x)^+)\Big) - \Big(R_1(x, A \cap B(x)^-) - m(A \cap B(x)^-)\Big).\end{aligned}$$

Then the operator norm on $L^1(X; m)$ determined by $|R_1(x, \cdot) - m(\cdot)|$ coincides with $\|R_1 - m\|_1$. By a similar argument using (10) instead of (5), we also have

$$\|R_1^n - m\|_\infty \leq 2\gamma^n. \tag{11}$$

Lemma 2.1 *Suppose that* **M** *satisfies the assumption (A). Then*

$$\int_X (u(x) - \langle m, u \rangle)^2 \, dm(x) \leq \frac{\gamma}{1 - \gamma} \mathcal{E}(u, u), \quad u \in \mathcal{F}. \tag{12}$$

In particular,

$$\|R_1 u - \langle m, u\rangle\|_2 \le \gamma \|u - \langle m, u\rangle\|_2. \tag{13}$$

Proof. Let $\lambda_1 > 0$ be the spectral gap of $(\mathcal{E}, \mathcal{F})$ as defined in §1. By (11) and the spectral representation,

$$\begin{aligned}\frac{1}{(1+\lambda_1)^{2n}} &\le \inf_{\|u\|_2=1, \langle m,u\rangle=0} \int_{\lambda_1}^{\infty} \frac{1}{(1+\lambda)^{2n}} dE_\lambda(u,u) \\ &= \|R_1^n - m\|_2^2 \le \|R_1^n - m\|_\infty^2 \\ &\le 4\gamma^{2n}.\end{aligned}$$

Therefore, $\lambda_1 \ge (2^{1/n}\gamma)^{-1} - 1$ for all $n \ge 1$. Now, letting $n \to \infty$, we get

$$\lambda_1 \ge \frac{1-\gamma}{\gamma}.$$

The inequalities (12) and (13) follow easily from this. $\square$

Let denote $(\cdot,\cdot)_\mu$ the inner product on $L^2(X;\mu)$. Define a potential kernel K by

$$Kf(x) = \sum_{n=1}^{\infty} \left(R_1^n f(x) - \langle m, f\rangle\right). \tag{14}$$

By virtue of (10), $Kf(x)$ is well defined for all $x \in X$ and satisfies

$$\|Kf\|_\infty \le 2\sum_{n=1}^{\infty} \gamma^n \|f\|_\infty = \frac{2\gamma}{1-\gamma}\|f\|_\infty$$

for any $f \in L^\infty(X;m)$. Moreover, by using (13),

$$(f, Kf)_m = \sum_{n=1}^{\infty} (\bar f, (R_1^n - m)\bar f)_m \le \sum_{n=1}^{\infty} \gamma^n \|\bar f\|_2^2 = \frac{\gamma}{1-\gamma}\|\bar f\|_2^2 \tag{15}$$

for any $f \in L^2(X;m)$. On the other hand, since

$$((R_1^n - m)f, g)_m = \int_0^\infty e^{-t} \left(\Phi_n f(t,\cdot), \Phi_n g(t,\cdot)\right)_m dt$$

for $\Phi_{2k} f(t,x) = (R_1^k - m)f(x)$ and $\Phi_{2k+1} f(t,x) = (R_1^k - m)p_{t/2} f(x)$, we have from (15)

$$(Kf, Kf)_m = \sum_{n=1}^{\infty} ((R_1^n - m)f, Kf)_m$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \int_0^{\infty} e^{-t} \left(\Phi_n f(t,\cdot), \Phi_n (Kf)(t,\cdot)\right)_m dt \\
&= \sum_{n=1}^{\infty} \int_0^{\infty} e^{-t} \left(\Phi_n f(t,\cdot), K\Phi_n f(t,\cdot)\right)_m dt \\
&\leq \frac{\gamma}{1-\gamma} \sum_{n=1}^{\infty} \int_0^{\infty} e^{-t} \left(\Phi_n f(t,\cdot), \Phi_n f(t,\cdot)\right)_m dt \\
&= \frac{\gamma}{1-\gamma} \left(f, Kf\right)_m . \qquad (16)
\end{aligned}
$$

Note that $Kf \in L^2(X;m)$ and satisfies

$$R_1 Kf = KR_1 f = Kf - R_1 f + \langle m, f\rangle, \quad f \in L^2(X;m).$$

Therefore, $\{Kf : f \in L^2(X;m)\} = \{R_1 f - \langle m, f\rangle : f \in L^2(X;m)\}$. Furthermore, Kf belongs to the domain of the generator $\mathcal{G}$ of $\mathbf{M}$ and

$$-\mathcal{G}Kf = -\mathcal{G}\left(R_1 Kf + R_1 f - \langle m, f\rangle\right) = f - \langle m, f\rangle.$$

In particular,

$$(f, Kf)_m = (f - \langle m, f\rangle, Kf)_m = -\left(\mathcal{G}Kf, Kf\right)_m = \mathcal{E}(Kf, Kf).$$

Put $K^{(0)}f := Kf + f - \langle m, f\rangle = \sum_{n=0}^{\infty}(R_1^n - m)f$. Similarly to (16), it holds that

$$\left(K^{(0)}f, K^{(0)}f\right)_m \leq \frac{1}{1-\gamma}\left(f, K^{(0)}f\right)_m . \qquad (17)$$

We say that $\mathbf{M}$ is recurrent in the sense of Harris if for any $F \subset X$ with $m(F) > 0$,

$$\int_0^{\infty} I_F(X_t)dt = \infty \quad \text{a.s. } P_x \text{ for all } x \in X.$$

In particular, if $\mathbf{M}$ is recurrent and $R_1(x,\cdot)$ is absolutely continuous relative to m for all $x \in X$, then $\mathbf{M}$ is recurrent in the sense of Harris ([2],[8]). In this case, as we stated in §1, the inequality (3) holds. Now, let us assume that $\mathbf{M}$ satisfies Harris recurrence condition to derive an L^2-version of (3).

Define for any positive continuous additive functional $A_t = \int_0^t I_C(X_s)ds$, the kernels R_A^{α} and K_A^{α} by

$$
\begin{aligned}
R_A^{\alpha} f(x) &= E_x\left(\int_0^{\infty} e^{-\alpha t - A_t} f(X_t)dt\right), \\
K_A^{\alpha} f(x) &= E_x\left(\int_0^{\infty} e^{-\alpha A_t} f(X_t)dA_t\right).
\end{aligned}
$$

In particular, put $R_A = R_A^0$. Under the present assumption of Harris recurence, for a set C with $m(C) > 0$, K_A^α is the resolvent of the recurrent time changed process on C by A_t. Furthermore, we can choose C satisfying $0 < m(C) < \infty$ and

$$\sup_{x\in C} \left\| K_A^1(x,\cdot) - \nu_C(\cdot) \right\| \le 2\gamma$$

for some $\gamma \in (0,1)$, where $\nu_C = (1/m(C))m|_C$ ([9],[11]). Similarly to (10), it then holds that

$$\left\| (K_A^1)^n(x,\cdot) - \nu_C(\cdot) \right\| \le 2\gamma^n, \quad \forall n \ge 1$$

and consequently, for any $f \in L^2(C;\nu_C)$,

$$\left\| (K_A^1)^n f - \langle \nu_C, f\rangle \right\|_2 \le \gamma^n \| f - \langle \nu_C, f\rangle \|_2$$

by virtue of Lemma 2.1. Put

$$K_A = \sum_{n=1}^{\infty} \left((K_A^1)^n - \nu_C \right), \quad K_A^{(0)} = \sum_{n=0}^{\infty} \left((K_A^1)^n - \nu_C \right).$$

From the symmetry,

$$\langle \nu_C, R_A f\rangle = \left(K_A^1 1, f \right)_m = \langle m, f\rangle.$$

Define a kernel K by

$$Kf = K_A^{(0)} R_A f = K_A^1 K_A^{(0)} R_A f + R_A f - \langle m, f\rangle.$$

By using the Markov property,

$$\begin{aligned} \alpha R_\alpha K_A^1 h &= R_\alpha (I_C \cdot K_A^1 h) + K_A^1 h - R_\alpha (I_C h), \\ \alpha R_\alpha R_A h &= R_\alpha \left(I_C \cdot R_A h \right) + R_A h - R_\alpha h. \end{aligned}$$

Since $K_A^1 K_A^{(0)} g = K_A^{(0)} g - g + \langle \nu_C, g\rangle$, we have

$$\begin{aligned} &\alpha R_\alpha K_A^1 K_A^{(0)} R_A f \\ &\quad = R_\alpha (I_C \cdot K_A^1 K_A^{(0)} R_A f) + K_A^1 K_A^{(0)} R_A f - R_\alpha (I_C \cdot K_A^{(0)} R_A f) \\ &\quad = R_\alpha (I_C \cdot K_A^{(0)} R_A f - I_C \cdot R_A f + I_C \cdot \langle \nu_C, R_A f\rangle) \\ &\qquad + K_A^1 K_A^{(0)} R_A f - R_\alpha (I_C \cdot K_A^{(0)} R_A f) \\ &\quad = -R_\alpha (I_C \cdot R_A f) + R_\alpha I_C \cdot \langle m, f\rangle + K_A^1 K_A^{(0)} R_A f. \end{aligned}$$

Hence it holds that

$$(I - \alpha R_\alpha) Kf = R_\alpha f - R_\alpha I_C \cdot \langle m, f\rangle \tag{18}$$

and consequently,

$$K\left(I-\alpha R_\alpha\right)f = R_\alpha f - \langle \nu_C, R_\alpha f\rangle. \tag{19}$$

Moreover, since

$$\begin{aligned}
\int_X \left(K_A^1 h\right)^2(x) g(x)\,dm(x) &\le \int_X K_A^1 h^2(x) g(x)\,dm(x)\\
&= \int_X R_A g(x) h^2(x)\,d\nu_C(x)\\
&\le \|R_A g\|_\infty \int_X h^2(x)\,d\nu_C(x),
\end{aligned}$$

applying (17) to $K_A^{(0)}|_{C\times C}$, we get for any bounded non-negative function $g \in L^1(X;m)$ satisfying $\|R_A g\|_\infty < \infty$ that

$$\begin{aligned}
\left(K_A R_A f, K_A R_A f\right)_{g\cdot m} &= \left\|K_A^1 K_A^{(0)} R_A f\right\|^2_{L^2(g\cdot m)}\\
&\le \|R_A g\|_\infty \left(K_A^{(0)} R_A f, K_A^{(0)} R_A f\right)_{\nu_C}\\
&\le \|R_A g\|_\infty \frac{1}{1-\gamma}\left(R_A f, K_A^{(0)} R_A f\right)_{\nu_C}\\
&= \|R_A g\|_\infty \frac{1}{1-\gamma}\left(f, K_A^1 K_A^{(0)} R_A f\right)_m\\
&= \|R_A g\|_\infty \frac{1}{1-\gamma}\left(f, K_A R_A f\right)_m.
\end{aligned}$$

Note that R_A is the potential kernel of the transient Dirichlet form $\mathcal{E}_A = \mathcal{E} + (\cdot,\cdot)_{\nu_C}$ on $L^2(X;m)$ and $R_A I_C = 1$. Thus we have from (2),

$$\begin{aligned}
&\int_X \left(R_A f - \langle m, f\rangle\right)^2(x) g(x)\,dm(x)\\
&\quad= \int_X \left(R_A(f - \langle m, f\rangle\cdot I_C)\right)^2(x) g(x)\,dm(x)\\
&\quad\le 2\|R_A g\|_\infty \mathcal{E}\left(R_A(f - \langle m, f\rangle\cdot I_C), R_A(f - \langle m, f\rangle\cdot I_C)\right)\\
&\quad\le 2\|R_A g\|_\infty \mathcal{E}_A\left(R_A(f - \langle m, f\rangle\cdot I_C), R_A(f - \langle m, f\rangle\cdot I_C)\right)\\
&\quad= 2\|R_A g\|_\infty \left(f - \langle m, f\rangle\cdot I_C, R_A(f - \langle m, f\rangle\cdot I_C)\right)_m\\
&\quad\le 2\|R_A g\|_\infty \left(f, R_A(f - \langle m, f\rangle\cdot I_C)\right)_m.
\end{aligned}$$

Therefore

$$\begin{aligned}
&(Kf, Kf)_{g\cdot m}\\
&\quad= \left(K_A R_A f + R_A f - \langle m, f\rangle, K_A R_A f + R_A f - \langle m, f\rangle\right)_{g\cdot m}\\
&\quad\le 2\left(K_A R_A f, K_A R_A f\right)_{g\cdot m} + 2\int_X \left(R_A f(x) - \langle m, f\rangle\right)^2 g(x)\,dm(x)
\end{aligned}$$

$$
\begin{aligned}
&\le 4\|R_A g\|_\infty \left\{ \frac{1}{1-\gamma} (f, K_A R_A f)_m + (f, R_A f - \langle m, f\rangle)_m \right\} \\
&\le \frac{4}{1-\gamma} \|R_A g\|_\infty (f, Kf)_m \\
&= \frac{4}{1-\gamma} \|R_A g\|_\infty \mathcal{E}(Kf, Kf). \qquad (20)
\end{aligned}
$$

On the other hand, from (18) and (19),

$$Kf = R_1 Kf + R_1(f - \langle m, f\rangle \cdot I_C)$$

and

$$R_1 f = Kf - KR_1 f + \langle \nu_C, R_1 f\rangle.$$

Thus the images of K and R_1 coincide except a difference of a constant factor which makes the integral by ν_C zero. Hence, we have from (20),

$$
\begin{aligned}
&\int_X |R_1 f(x) - \langle \nu_C, R_1 f\rangle|^2 g(x)\, dm(x) \\
&\quad = \int_X |K(I - R_1) f(x)|^2 g(x)\, dm(x) \\
&\quad \le \frac{4}{1-\gamma} \|R_A g\|_\infty \mathcal{E}(K(I-R_1)f, K(I-R_1)f) \\
&\quad = \frac{4}{1-\gamma} \|R_A g\|_\infty \mathcal{E}(R_1 f, R_1 f).
\end{aligned}
$$

Further, approximating $u \in \mathcal{F}$ by a sequence of the form $u_n = R_1 f_n$ such that $u_n \to u$ in $\mathcal{F}$, we get the following result.

Theorem 2.1 *If* $\mathbf{M}$ *is recurrent in the sense of Harris, then there exists a set* C *such that*

$$\frac{1}{m(C)} \int_X |u(x) - \langle \nu_C, u\rangle|^2 g(x)\, dm(x) \le \frac{4\|R_A g\|_\infty}{1-\gamma} \mathcal{E}(u, u) \qquad (21)$$

for any $u \in \mathcal{F}$ *and a bounded non-negative function* $g \in L^1(X; m)$ *such that* $\|R_A g\|_\infty < \infty$.

3 Transience and recurrence of sets relative to certain time inhomogeneous diffusion processes

In this section, we assume that we are given an irreducible m-symmetric diffusion process $\mathbf{M} = (X_t, P_x)$ on X which is associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ given by

$$\mathcal{E}(\varphi, \psi) = \frac{1}{2} \int_X d\mu_{\langle \varphi, \psi\rangle}(x). \qquad (22)$$

We consider that the path space is canonical and $X_t(\omega) = \omega(t)$. Denote the associated generator by $\mathcal{G}$. Fix a strictly positive continuous function $\rho(t,x)$ such that $\rho(t,\cdot) \in \mathcal{F}$ and $t \mapsto \partial\rho(t,\cdot)/\partial t$ is a measurable function from $[0,\infty)$ to $L^2_{loc}(X;m)$. Put $\mu_t(dx) = \rho^2(t,x)m(dx)$ and consider the Dirichlet form $(\mathcal{E}^{(t)}, \mathcal{F}^{(t)})$ on $L^2(X;\mu_t)$ determined by

$$\mathcal{E}^{(t)}(\varphi,\psi) = \frac{1}{2}\int_X \rho^2(t,x)\, d\mu_{\langle\varphi,\psi\rangle}(x). \tag{23}$$

Denote by $\mathcal{G}^{(t)}$ the generator corresponding to $(\mathcal{E}^{(t)}, \mathcal{F}^{(t)})$. A time inhomogeneous diffusion process $\mathbf{M}^\rho = (X_t, P^\rho_{(s,x)})$ is said associated with the family of Dirichlet forms $(\mathcal{E}^{(t)}, \mathcal{F}^{(t)})$ if its transition function $u_t(s,x) = E^\rho_{(s,x)}(\varphi(X_{t-s}))$ satisfies the terminal value problem

$$\frac{\partial u_t(s,x)}{\partial s} + \mathcal{G}^{(s)}u_t(s,x) = 0, \quad u_t(t,x) = \varphi(x), \tag{24}$$

for $s < t$. Denote by R^ρ_α the resolvent of $\mathbf{M}^\rho$, that is

$$R^\rho_\alpha\varphi(s,x) = E^\rho_{(s,x)}\left(\int_0^\infty e^{-\alpha t}\varphi(X_t)\, dt\right).$$

Then (24) is equivalent to

$$-\left(\frac{\partial R^\rho_\alpha\varphi(s,\cdot)}{\partial s}, \psi\right)_{\mu_s} + \mathcal{E}^{(s)}_\alpha\left(R^\rho_\alpha\varphi(s,\cdot),\psi\right) = (\varphi,\psi)_{\mu_s} \tag{25}$$

for any $\varphi \in \mathcal{F}^{(s)}$.

There also exists a diffusion process $\widehat{\mathbf{M}}^\rho = (X_s, \widehat{P}^\rho_{(t,y)})$ which is a dual process of $\mathbf{M}^\rho$ in the sense

$$\int_X E^\rho_{(s,x)}\left(\psi\left(X_{t-s}\right)\right)\varphi(x)\, d\mu_s(x) = \int_X \widehat{E}^\rho_{(t,y)}\left(\varphi(X_{t-s})\right)\psi(y)\, d\mu_t(y) \tag{26}$$

for any $\varphi,\psi \geq 0$. Note that the measure $\widehat{P}^\rho$ is not necessarily sub-Markov. In fact, $\widehat{P}^\rho$ is given by the following transformation by a multiplicative functional

$$\widehat{P}^\rho_{(t,y)}(\Lambda) = E_y\left(\exp\left(M^{[\log\rho]}_{t-s} - \frac{1}{2}\langle M^{[\log\rho]}\rangle_{t-s}\right) e^{-\widehat{B}_{t-s}} : \Lambda\right),$$

for $\Lambda \in \sigma(X_\tau; \tau \leq t-s)$, where $M^{[\log\rho]}_\tau$ is the martingale part appearing in the decomposition

$$\log\rho(t-\tau, X_\tau) - \log\rho(t, X_0) = M^{[\log\rho]}_\tau + N^{[\log\rho]}_\tau$$

into a martingale additive functional of finite energy and a continuous additive functional of zero energy relative to $\widehat{P}^\rho_{(t,y)}$, and

$$\widehat{B}_s = \int_0^s \frac{\partial\log\rho}{\partial t}(t-\tau, X_\tau)d\tau.$$

Hence, we have

$$\hat{P}^{\rho}_{(t,y)}(X) \leq \exp\left(\ell_t(s)\right), \quad \ell_t(s) = \int_0^{t-s} \left\| \frac{\partial \log \rho}{\partial t}(t-\tau, \cdot) \right\|_\infty d\tau. \tag{27}$$

Fix a closed set F of X such that $\rho^2(t,\cdot) \in L^1(D;m)$ for $D = X \setminus F$. By considering $\rho^2(t,x)/Z(t)$ instead of $\rho^2(t,x)$, we may assume that $\mu_t(dx) = \rho^2(t,x)m(dx)$ is a probability measure on D, where

$$Z(t) = \int_D \rho^2(t,x)dm(x).$$

Let f be a non-negative function on D such that $\langle \mu_s, f \rangle = 1$ for fixed $s \geq 0$. For such f, define the function $\widehat{u}^D_s$ by

$$\widehat{u}^D_s(t,y) = \widehat{E}^{\rho}_{(t,y)}\left(f(X_{t-s}) : t-s < \sigma_F\right), \quad y \in D \tag{28}$$

and put

$$\widehat{H}^D_s(t) = \int_D (\widehat{u}^D_s)^2(t,y)\, d\mu_t(y).$$

We assume that the number

$$\lambda_D(t) = -\inf_{x \in D} \frac{\partial}{\partial t} \log \rho^2(t,x)$$

is finite. For instance, this assumption holds if D is relatively compact. Then we have the following lemma relative to $\widehat{H}^D_s(t)$.

Lemma 3.1 (i) *For any $s < t$,*

$$\mathcal{E}^{(t)}(\widehat{u}^D_s(t,\cdot), \widehat{u}^D_s(t,\cdot)) \leq -\frac{1}{2}\frac{d}{dt}\widehat{H}^D_s(t) + \frac{1}{2}\lambda_D(t)\widehat{H}^D_s(t).$$

(ii) *If $\lim_{t\to\infty} \widehat{H}^D_s(t) = 0$, then $P^{\rho}_{f\cdot\mu_s}(\sigma_F < \infty) = 1$.*

Proof. (i) Since $\widehat{u}^D_s$ satisfies

$$\frac{1}{\rho^2(t,y)} \frac{\partial(\rho^2 \widehat{u}^D_s)}{\partial t}(t,y) = \widehat{\mathcal{G}}^{(t)}\widehat{u}^D_s(t,y) \tag{29}$$

with condition

$$\widehat{u}^D_s(s,y) = f(y), \quad \widehat{u}^D_s(t,y) = 0 \ \text{ for } y \in F,$$

by multiplying $\widehat{u}^D_s(t,y)$ and integrating on D by $d\mu_t(y)$, we have

$$\int_D \frac{\partial(\rho^2 \widehat{u}^D_s(t,y))}{\partial t} \widehat{u}^D_s(t,y)\, dm(y) = -\mathcal{E}^{(t)}\left(\widehat{u}^D_s(t,\cdot), \widehat{u}^D_s(t,\cdot)\right). \tag{30}$$

Since the lefthand side of (30) can be written as

$$\frac{1}{2}\frac{d}{dt}\int_D (\widehat{u}_s^D)^2(t,y)\rho^2(t,y)\,dm(y) + \frac{1}{2}\int_D (\widehat{u}_s^D)^2(t,y)\frac{\partial \rho^2(t,y)}{\partial t}\,m(dy)$$

we get the result.

(ii) By virtue of the duality relation (26), it holds that

$$\begin{aligned} P^\rho_{f\cdot\mu_s}(t-s<\sigma_F) &= E^\rho_{\mu_s}(f(X_0)I_D(X_{t-s}) : t-s<\sigma_F) \\ &= \widehat{E}^\rho_{\mu_t}(f(X_{t-s})I_D(X_0) : t-s<\sigma_F) \\ &= \int_D \widehat{u}_s^D(t,y)\,d\mu_t(y) \\ &\le \sqrt{\widehat{H}_s^D(t)}. \end{aligned}$$

Hence, we have the assertion. □

Assume that F is a non-exceptional closed set. By virtue of the irreducibility of $\mathbf{M}$, its part process $\mathbf{M}_D$ on D is transient. Hence, applying (2) for $\mathbf{M}_D$, for any bounded positive function $g \in L^1(X;m)$ such that $\|R^D g\|_\infty < \infty$,

$$\int_D u^2(x)g(x)dm(x) \le 2\|R^D g\|_\infty\, \mathcal{E}(u,u) \tag{31}$$

for all $u \in \mathcal{F}_D = \{u \in \mathcal{F} : \tilde{u} = 0 \text{ q.e. on } F\}$. If $m(D) < \infty$ and $\|R^D 1\|_\infty < \infty$, then (31) holds for $g = 1$. As a typical case, this holds if D is compact and the transition function p_t of $\mathbf{M}$ is strong Feller. In fact, it then holds that $\inf_{x\in D} p_t(x,F) > 0$ and

$$p_t^D 1(x) \le 1 - p_t(x,F) \le 1 - \inf_{x\in D} p_t(x,F) < 1$$

for any $x \in D$, where p_t^D is the transition function of $\mathbf{M}_D$.

Now, we give a general criterion on ρ for the recurrence of the set F relative to $\mathbf{M}^\rho$. Put

$$\delta_D(t) = \frac{\inf_D \rho^2(t,\cdot)}{\sup_D(\rho^2(t,\cdot)/g(\cdot))}.$$

Since g is bounded, $\delta_D(t) < \infty$.

Theorem 3.1 *Suppose that there exists a positive function $g \in L^1(D;m)$ such that*

$$\lim_{T\to\infty}\int_s^T \left(\lambda_D(t) - \left\|R^D g\right\|_\infty^{-1}\delta_D(t)\right)dt = -\infty. \tag{32}$$

Then $P^\rho_{f\cdot\mu_s}(\sigma_F<\infty) = 1$ for any non-negative function f with $\langle\mu_s, f\rangle = 1$. In particular, if the transition density of $\mathbf{M}^\rho$ exists, then $P^\rho_{(s,x)}(\sigma_F<\infty) = 1$ for all $x \in D$.

Proof. Let $g > 0$ be a function satisfying the stated condition. Then we have from (31)

$$\begin{aligned}\widehat{H}_s^D(t) &\le 2\sup_{x\in D}(\rho^2(t,x)/g(x))\,\|R^D g\|_\infty\,\mathcal{E}\left(\widehat{u}_s^D(t,\cdot),\widehat{u}_s^D(t,\cdot)\right)\\ &\le 2\|R^D g\|_\infty\,\delta_D^{-1}(t)\,\mathcal{E}^{(t)}\left(\widehat{u}_s^D(t,\cdot),\widehat{u}_s^D(t,\cdot)\right).\end{aligned}$$

Combining this with the result (i) of Lemma 3.1, we get that

$$\left\|R^D g\right\|_\infty^{-1}\delta_D(t)\,\widehat{H}_s^D(t) \le -\frac{d}{dt}\widehat{H}_s^D(t) + \lambda_D(t)\widehat{H}_s^D(t),$$

that is,

$$\frac{d}{dt}\log\widehat{H}_s^D(t) \le \left(\lambda_D(t) - \left\|R^D g\right\|_\infty^{-1}\delta_D(t)\right).$$

Hence, we have

$$\widehat{H}_s^D(T) \le \widehat{H}_s^D(s)\exp\left(\int_s^T\left(\lambda_D(t) - \left\|R^D g\right\|_\infty^{-1}\delta_D(t)\right)dt\right). \tag{33}$$

Therefore the first assertion follows from Lemma 3.1 (ii). If the transition density $p^\rho(s,x;t,y)$ of $\mathbf{M}^\rho$ exists, then

$$\begin{aligned}P^\rho_{(s,x)}(t-s<\sigma_F) &= P^\rho_{(s,x)}(\tau-s<\sigma_F, t-\tau<\sigma_F\circ\theta_{\tau-s})\\ &= P^\rho_{f\cdot\mu_\tau}(t-\tau<\sigma_F)\to 0,\quad t\to\infty\end{aligned}$$

for $f(y) = p^\rho(s,x;\tau,y)$. □

Example 3.1 Suppose that X is a Riemannian manifold with volume element m and $\mathbf{B} = (X_t, P_x)$ the Brownian motion on X. Then the associated Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(X;m)$ is given by

$$\mathcal{E}(\varphi,\psi) = \frac{1}{2}\int_X \nabla\varphi(x)\cdot\nabla\psi(x)\,dm(x).$$

Let U be a smooth locally bounded non-negative function on X and $\rho(t,x)$ a function defined by

$$\rho(t,x) = \exp\left(-\frac{1}{2}\beta(t)U(x)\right). \tag{34}$$

for $\beta(t) = (1/c)\log(1+t), c > 0$. We consider the associated time inhomogeneous diffusion process $\mathbf{B}^\rho = (X_t, P^\rho_{(s,x)})$ for the function (34).

Fix a connected component D of a level set of the form $\{x : U(x) \le b\}$ and let $a = \inf_D U$. To make $\mu_t(D) = 1$, we consider $\rho^2(t,x)/Z(t)$ instead of $\rho^2(t,x)$, that is we consider as

$$\mu_t(dy) = \frac{\rho^2(t,y)m(dy)}{Z(t)}, \quad Z(t) = \int_D \rho^2(t,y)dm(y).$$

Since D is compact, it is easy to see that the part process $\mathbf{B}_D$ of $\mathbf{B}$ on D satisfies (31) for $g = 1$. By elementary calculations, since

$$\lambda_D(t) = \frac{b}{c(1+t)} + \frac{d}{dt}\log Z(t), \quad \text{and} \quad \delta_D(t) = (1+t)^{-\frac{(b-a)}{c}}, \tag{35}$$

we have

$$\begin{aligned}
\int_s^T \lambda_D(t)dt &= \frac{b}{c}\log\frac{1+T}{1+s} + \log\frac{Z(T)}{Z(s)} \\
\int_s^T \delta_D(t)dt &= \frac{c}{c-(b-a)}\left((1+T)^{1-(b-a)/c} - (1+s)^{1-(b-a)/c}\right)
\end{aligned}$$

and for any $\varepsilon > 0$,

$$m(\{a < U < a+\varepsilon\})(1+t)^{-\frac{a+\varepsilon}{c}} \le Z(t) \le m(D)(1+t)^{-\frac{a}{c}}.$$

Therefore, if $b - a < c$,

$$\lim_{T\to\infty}\int_s^T \left(\lambda_D(t) - 2\left\|R^D 1\right\|_\infty^{-1}\delta_D(t)\right)dt = -\infty$$

and which implies that the set $F = X \setminus D$ is a recurrent set relative to $\mathbf{B}^\rho$ by virtue of Theorem 3.1.

Next, we turn to a general condition on ρ for the transience of some sets relative to $\mathbf{M}^\rho$. We assume that the state space X is compact and $\mathbf{M}$ is recurrent on it. Before considering time inhomogeneous process $\mathbf{M}^\rho$, we give an estimation of the type (12) relative to $\mathbf{M}^{\bar\rho}$ for time independent $\bar\rho$.

Fix a positive function $\bar\rho(x) \in \mathcal{D}(\mathcal{G})$ such that $\mu_{\log\bar\rho}(x) := d\mu_{\langle\log\bar\rho^2,\log\bar\rho^2\rangle}(x)/dm(x) < \infty$ and $\|\mathcal{G}\log\bar\rho\|_\infty < \infty$. Put $d\bar\mu(x) = \bar\rho^2(x)dm(x)$. Let $(\mathcal{E}^{\bar\rho}, \mathcal{F}^{\bar\rho})$ be the Dirichlet form on $L^2(X;\bar\mu)$ determined by

$$\mathcal{E}^{\bar\rho}(\varphi,\psi) = \frac{1}{2}\int_X \bar\rho^2(x)d\mu_{\langle\varphi,\psi\rangle}(x)$$

and $\mathbf{M}^{\bar\rho} = (X_t^{\bar\rho}, P_x^{\bar\rho})$ the associated diffusion process with resolvent $R_\alpha^{\bar\rho}$. Put $h(x) = \mathcal{G}\log\bar\rho(x) + 2\mu_{\log\bar\rho}(x)$, $C_t = \int_0^t h^+(X_s)ds$ and

$$R_C^\alpha f(x) = E_x\left(\int_0^\infty e^{-\alpha t - C_t} f(X_t)dt\right).$$

Then it satisfies

$$R_C^\alpha f(x) = R_\alpha f(x) - R_C^\alpha (h \cdot R_\alpha f)(x) \tag{36}$$

(§4.6 in [4]). If the density $R_\alpha(x,y)$ of $R_\alpha(x,dy)$ relative to $m(dy)$ exists, then (36) implies that the density of $R_C^\alpha(x,\cdot)$ relative to m also exists. We denote its density by $R_C^\alpha(x,y)$.

For convenience, we will assume $\sup_{x\in X} \bar\rho(x) = 1$ with no loss in generality. Put

$$H^{\bar\rho}(x,y) = \inf_\eta \sup_{0\le t\le 1} \left(-\log \bar\rho^2(\eta(t))\right), \quad x,y \in X,$$

where $\eta(t), 0 \le t \le 1$ is a curve in X connecting x and y. Furthermore, put

$$\begin{aligned} m(\bar\rho) &= \sup_{x,y\in X} \left\{H^{\bar\rho}(x,y) + \log \bar\rho^2(x) + \log \bar\rho^2(y)\right\}, \\ \gamma(\bar\rho) &= 1 - e^{-m(\bar\rho)/2} \inf_{z,w} R_C^1(z,w). \end{aligned}$$

Lemma 3.2 *Suppose that $R_\alpha(x,dy)$ has a strictly positive lower semi-continuous density $R_\alpha(x,y)$ relative to $\bar\mu(dy)$. Then*

$$\int_X (u(x) - \langle \bar\mu, u\rangle)^2 \, d\bar\mu(x) \le \frac{\gamma(\bar\rho)}{1-\gamma(\bar\rho)} \mathcal{E}^{\bar\rho}(u,u) \tag{37}$$

for any $u \in \mathcal{F}^{\bar\rho}$.

Proof. For any fixed $x \in X$ and $\varepsilon > 0$, let $\{B_i\}$ be a covering of X of finite open sets such that $|H^{\bar\rho}(x,z) - H^{\bar\rho}(x,w)| < \epsilon$ for all $z, w \in B_i$. From the definition of $H^{\bar\rho}$, the process $(X_\tau)_{0\le\tau\le t}$ hits the set $\{y : \rho^2(y) \le \exp(-H^{\bar\rho}(x,X_t))\}$ $P_x^{\bar\rho}$-a.s.. Thus, for any i, if we take a point $z_i \in B_i$, then for $P_x^{\bar\rho}$-a.s. ω such that $X_t(\omega) \in B_i$, $(X_\tau)_{0\le\tau\le t}$ hits

$$B_{x,i} = \left\{y : \bar\rho^2(y) < \exp\left(-H^{\bar\rho}(x,z_i) + \varepsilon\right)\right\}.$$

Hence if we denote by $\sigma_{x,i}$ the hitting time of $\mathbf{M}^{\bar\rho}$ to $B_{x,i}$, then from the continuity of $\bar\rho^2$,

$$\begin{aligned} & R_1^{\bar\rho} I_{B_i}(x) \\ &\quad = E_x\left(\int_0^\infty \frac{\bar\rho(X_t)}{\bar\rho(x)} e^{-t-C_t} I_{B_i}(X_t)\, dt\right) \\ &\quad \ge E_x\left(\int_{\sigma_{x,i}}^\infty \frac{\bar\rho(X_{\sigma_{x,i}})}{\bar\rho(X_t)\bar\rho(x)} \, \frac{1}{\bar\rho(X_{\sigma_{x,i}})} \, e^{-t-C_t} \, \bar\rho^2(X_t) I_{B_i}(X_t)\, dt\right) \\ &\quad = E_x\left(\int_{\sigma_{x,i}}^\infty e^{-\frac{1}{2}\left(H^{\bar\rho}(x,z_i) - \varepsilon + \log \bar\rho^2(x) + \log \bar\rho^2(X_t)\right)} \frac{1}{\bar\rho(X_{\sigma_{x,i}})}\right. \end{aligned}$$

$$
\begin{aligned}
&\qquad \times\, e^{-t-C_t}\ \bar\rho^2(X_t) I_{B_i}(X_t)\,dt\Big) \\
&\geq E_x\left(\int_{\sigma_{x,i}}^\infty e^{-\frac{1}{2}\left(H^{\bar\rho}(x,X_t)+\log\bar\rho^2(x)+\log\bar\rho^2(X_t)\right)}\frac{1}{\bar\rho(X_{\sigma_{x,i}})}\right. \\
&\qquad \times\, e^{-t-C_t}\ \bar\rho^2(X_t) I_{B_i}(X_t)\,dt\Big) \\
&\geq e^{-m(\bar\rho)/2} E_x\left(\int_{\sigma_{x,i}}^\infty e^{-t-C_t}\ \bar\rho^2(X_t) I_{B_i}(X_t)\,dt\right) \\
&= e^{-m(\bar\rho)/2}\ R_C^1(\rho^2 I_{B_i})(x).
\end{aligned}
$$

This implies that for any $x, y \in X$,

$$
R_1^{\bar\rho}(x,y) \geq e^{-m(\bar\rho)/2}\ R_C^1(x,y).
$$

Therefore, by putting $\Gamma_{x,y}^+ = \{z : R_1^{\bar\rho}(x,z) - R_1^{\bar\rho}(y,z) > 0\}$, we have

$$
\begin{aligned}
\left\| R_1^{\bar\rho}(x,\cdot) - R_1^{\bar\rho}(y,\cdot)\right\| &= 2\left(R_1^{\bar\rho}(x,\Gamma_{x,y}^+) - R_1^{\bar\rho}(y,\Gamma_{x,y}^+)\right) \\
&= 2\int_{\Gamma_{x,y}^+}\left(R_1^{\bar\rho}(x,z) - R_1^{\bar\rho}(y,z)\right) d\bar\mu(z) \\
&\leq 2\int_X \left(R_1^{\bar\rho}(x,z) - \inf_{y,w} R_1^{\bar\rho}(y,w)\right) d\bar\mu(z) \\
&\leq 2\gamma(\bar\rho).
\end{aligned}
$$

This implies

$$
\left\|(R_1^{\bar\rho})^n(x,\cdot) - \bar\mu(\cdot)\right\| \leq 2\gamma(\bar\rho)^n
$$

for all $n \geq 1$. Hence the result follows from Lemma 2.1. □

Now we return to the time inhomogeneous case. Put $m(t) = m(\rho(t,\cdot))$ and $\gamma(t) = \gamma(\rho(t,\cdot))$. Let $\widehat{u}_s^D(t,x)$ and $\lambda_D(t)$ are those defined before Lemma 3.1. We omit the superfix D if $D = X$. Put

$$
\widehat{V}_s(t) = \|\widehat{u}_s(t,\cdot) - \langle \mu_t, \widehat{u}_s(t,\cdot)\rangle\|_{L^2(\mu_t)}^2.
$$

Lemma 3.3 *Assume the condition of Lemma 3.2. Then, for any $s \geq 0$ and a non-negative function f such that $\langle \mu_s, f\rangle = 1$,*

$$
\widehat{V}_s(T) \leq e^{-\int_s^T S_{\gamma,\lambda}(t)\,dt}\left\{\|f-1\|_{L^2(\mu_s)}^2 + \int_s^T \lambda(t) e^{\int_s^t S_{\gamma,\lambda}(\tau)\,d\tau} dt\right\},
$$

where

$$
S_{\gamma,\lambda}(t) = \frac{1-\gamma(t)}{\gamma(t)} - \lambda(t).
$$

Proof. Note that it can be written as $\widehat{V}_s(t) = \|\widehat{u}_s(t,\cdot) - 1\|^2_{L^2(\mu_t)} = \|\widehat{u}_s(t,\cdot)\|^2_{L^2(\mu_t)} - 1$. Then, $\widehat{V}_s(t)$ satisfies

$$\mathcal{E}^{(t)}\left(\widehat{u}_s(t,\cdot), \widehat{u}_s(t,\cdot)\right) \leq -\frac{1}{2}\frac{d}{dt}\widehat{V}_s(t) + \frac{1}{2}\lambda(t)\widehat{V}_s(t) + \frac{1}{2}\lambda(t)$$

by virtue of Lemma 3.1. Using (37), a similar argument as the proof of Theorem 3.1 gives the the assertion of the lemma. □

Theorem 3.2 *Suppose that $R_\alpha(x,dy)$ has a strictly positive lower semi-continuous density $R_\alpha(x,y)$ and*

$$\limsup_{t\to\infty} \frac{\lambda(t)\gamma(t)}{1-\gamma(t)} < 1. \tag{38}$$

Moreover, assume a bounded non-negative function φ satisfies the following conditions ; for each $t > 0$, there exists $k(t)$ such that $0 < k(t) < t$, $k(t) \nearrow \infty$ as $t \nearrow \infty$,

$$\int_0^\infty e^{\ell_t(k(t))}\langle \mu_t, \varphi\rangle dt < \infty \tag{39}$$

and

$$\int_0^\infty \|\varphi\|_{L^2(\mu_t)} e^{-\frac{1}{2}\int_{k(t)}^t S_{\gamma,\lambda}(\tau)\,d\tau} dt < \infty. \tag{40}$$

Then $\int_0^\infty E^\rho_{f\cdot\mu_0}(\varphi(X_t))\,dt < \infty$ for any non-negative function f with $\langle \mu_0, f\rangle = 1$. In particular, if the transition density of $\mathbf{M}^\rho$ exists, then $\int_0^\infty E^\rho_{(0,x)}(\varphi(X_t))\,dt < \infty$.

Proof. For a function φ satisfying the stated conditions and a non-negative function f such that $\langle \mu_0, f\rangle = 1$,

$$\begin{aligned} E^\rho_{f\cdot\mu_0}(\varphi(X_t)) &= \int_X (\hat{u}_0(s,y) - 1)u_t(s,y)\,d\mu_s(y) + \langle \mu_s, u_t(s,\cdot)\rangle \\ &\leq \sqrt{\widehat{V}_0(s)}\sqrt{H_t(s)} + \langle \mu_s, u_t(s,\cdot)\rangle, \end{aligned} \tag{41}$$

where $u_t(s,x) = E^\rho_{(s,x)}(\varphi(X_{t-s}))$ and $H_t(s) = \|u_t(s,\cdot)\|^2_{L^2(\mu_s)}$. Since $u_t(s,\cdot)$ satisfies (24), we have from (37) that

$$\begin{aligned} \frac{d}{ds}H_t(s) &= 2\mathcal{E}^{(s)}(u_t(s,\cdot), u_t(s,\cdot)) + \int_X u_t^2(s,x)\frac{\partial}{\partial s}\log\rho^2(s,x)\,d\mu_s(x) \\ &\geq \frac{1-\gamma(s)}{\gamma(s)}\left(H_t(s) - \langle \mu_s, u_t(s,\cdot)\rangle^2\right) - \lambda(s)H_t(s). \end{aligned}$$

Therefore,

$$\begin{aligned} H_t(s) \leq\ & e^{-\int_s^t S_{\gamma,\lambda}(\tau)\,d\tau} \\ & \times\left\{\|\varphi\|^2_{L^2(\mu_t)} + \int_s^t e^{\int_\sigma^t S_{\gamma,\lambda}(\tau)\,d\tau}\frac{1-\gamma(\sigma)}{\gamma(\sigma)}\langle \mu_\sigma, u_t(\sigma,\cdot)\rangle^2 d\sigma\right\}. \end{aligned}$$

By virtue of (26) and (27), it holds that

$$\langle \mu_s, u_t(s,\cdot)\rangle = \int_X \varphi(y)\hat{P}^{\rho}_{(t,y)}\left(X_{t-s} \in X\right)\, d\mu_t(y) \le e^{\ell_t(s)}\langle \mu_t, \varphi\rangle. \tag{42}$$

From (38), we see that there exists a positive constant k_0 such that

$$\frac{1-\gamma(t)}{\gamma(t)} \le k_0 S_{\gamma,\lambda}(t)$$

for all t. Then, noting that $\ell_t(\sigma) \le \ell_t(s)$ for all $\sigma > s$,

$$\begin{aligned}
H_t(s) &\le e^{-\int_s^t S_{\gamma,\lambda}(\tau)\,d\tau} \\
&\quad \times \left\{ \|\varphi\|^2_{L^2(\mu_t)} + \int_s^t e^{\int_\sigma^t S_{\gamma,\lambda}(\tau)\,d\tau} \frac{1-\gamma(\sigma)}{\gamma(\sigma)} e^{2\ell_t(\sigma)}\langle \mu_t, \varphi\rangle^2 d\sigma \right\} \\
&\le \|\varphi\|^2_{L^2(\mu_t)} e^{-\int_s^t S_{\gamma,\lambda}(\tau)\,d\tau} \\
&\quad + k_0 e^{2\ell_t(s)}\langle \mu_t, \varphi\rangle^2 e^{-\int_s^t S_{\gamma,\lambda}(\tau)\,d\tau}\left(e^{\int_s^t S_{\gamma,\lambda}(\tau)\,d\tau} - 1\right).
\end{aligned}$$

Hence, by taking $s = k(t)$ for large t, we have

$$\sqrt{H_t(k(t))} \le e^{-\frac{1}{2}\int_{k(t)}^t S_{\gamma,\lambda}(\tau)\,d\tau}\|\varphi\|_{L^2(\mu_t)} + \sqrt{k_0} e^{\ell_t(k(t))}\langle \mu_t, \varphi\rangle. \tag{43}$$

Similarly, since

$$e^{-\int_0^t S_{\gamma,\lambda}(\tau)\,d\tau}\left\{ \|f-1\|^2_{L^2(\mu_0)} + \int_0^t \lambda(\tau) e^{\int_0^\tau S_{\gamma,\lambda}(\sigma)\,d\sigma} d\tau \right\} \le \|f-1\|^2_{L^2(\mu_0)} + 1,$$

$\widehat{V}_0(k(t))$ is bounded relative to t by virtue of Lemma 3.3. Therefore, we see that the first term of the righthand side of (41) is integrable relative to t from (39), (40) and (43). The second term of the righthand side of (41) is also integrable because (42) holds for $s = k(t)$. The last assertion of the theorem can be proved by the Markov property used in the proof of Theorem 3.1. □

Note that the result of Theorem 3.2 also holds for any time dependent function $\varphi(t,x)$ if it satisfies the conditions (39) and (40).

In the next corollary, we assume for any open set G and closed subset K of G, there exists a function $\varphi \in \mathcal{D}(\mathcal{G})$ such that $\varphi = 0$ on $X \setminus G$, $\varphi = 1$ on K and $\mu_\varphi(x) = d\mu_{\langle\varphi,\varphi\rangle}(x)/dm(x)$ is bounded.

Corollary 3.1 *Assume that the conditions of Theorem 3.2 and the conditions stated above. Let F be a closure of a non-empty open set of X such that $\sqrt{I_F \mu_{\log\rho(t,\cdot)}} \vee I_F$ satisfies (39) and (40). Then*

$$\lim_{T\to\infty} P^{\rho}_{(0,x)}\left(X_t \in F \ for\ some\ t \ge T\right) = 0,$$

that is, F is a transient set relative to $\mathbf{M}^\rho$.

Proof. The proof is similar to Lemma 3.2 in Holley et. al. ([6]). Let F_1 be a closed subset of the interior of F and $\varphi \in \mathcal{D}(\mathcal{G})$ a function such that $\varphi = 0$ on $X \setminus F$, $\varphi = 1$ on F_1 and μ_φ is bounded. Let $\mu_{t,\varphi}(\cdot)$ be the density of $d\mu_{\langle \log \rho(t,\cdot),\varphi\rangle}(\cdot)$ relative to $dm(\cdot)$. Then

$$M_t = \varphi(X_t) - \int_0^t (\mathcal{G}\varphi(X_\tau) + \mu_{t,\varphi}(X_\tau))\, d\tau$$

is a $P^\rho_{(0,x)}$-martingale. By virtue of Theorem 3.2, it holds that

$$\int_0^\infty E^\rho_{(0,x)}(|\mathcal{G}\varphi|(X_t))\, dt \le \|\mathcal{G}\varphi\|_\infty \int_0^\infty E^\rho_{(0,x)}(I_F(X_t)dt) < \infty.$$

Moreover, since

$$|\mu_{t,\varphi}|(x) \le \sqrt{\|\mu_\varphi\|_\infty}\, \sqrt{I_F(x)\mu_{\log \rho(t,\cdot)}(x)},$$

Theorem 3.2 also implies

$$\int_0^\infty E^\rho_{(0,x)}(|\mu_{t,\varphi}|(X_t))\, dt < \infty.$$

Hence, the martingale convergence theorem implies that M_t and hence $\varphi(X_t)$ converges to zero a.s. and the assertion of the corollary holds. □

Example 3.2 We consider the function ρ in Example 3.1 on a compact connected Riemannian manifold X. For simplicity, let $H_U(x,y) = \inf_\eta \sup_{0\le\tau\le 1} U(\eta(\tau))$ for a curve $\eta(\tau)$ connecting x and y, and assume that $\inf_X U = 0$. Put

$$m_U = \sup_{x,y\in X} (H(x,y) - U(x) - U(y)).$$

Then $m(t) = \beta(t)m_U - \log Z(t)$. Note that for any fixed t,

$$\begin{aligned} R^1_C f(x) &= E_x\left(\int_0^\infty e^{-\tau - \int_0^\tau \left(-\frac{1}{4}\beta(t)\Delta U + \frac{1}{8}\beta^2(t)|\nabla U|^2\right)^+ (X_\sigma)\, d\sigma} f(X_\tau)\, d\tau\right) \\ &\ge R_{1+\alpha(t)} f(x) \end{aligned}$$

for $\alpha(t) = \frac{1}{4}\,\beta(t)\|\Delta U\|_\infty + \frac{1}{8}\beta^2(t)\|\nabla U\|^2_\infty$. Put

$$\Gamma(t) = \inf_{z,w} R_{1+\alpha(t)}(z,w).$$

Since $Z(t) \ge m(\{x : U(x) < \varepsilon_1\})(1+t)^{-\varepsilon_1/c}$, we have

$$\begin{aligned} \gamma(t) &\le 1 - e^{-m(t)/2} Z(t)\Gamma(t) \\ &\le 1 - k(\varepsilon_1)(1+t)^{-(m_U/2c+\varepsilon_1)}\, \Gamma(t). \end{aligned}$$

Thus, if

$$\Gamma(t) \ge k_0(1+t)^{-\delta} \tag{44}$$

for some constant k_0 and $\delta < 1 - (m_U/2c)$, then (38) holds. Moreover, since

$$\ell_t(s) = \int_0^s \frac{\|U\|_\infty}{2c} \frac{1}{1+t-\tau} d\tau = \frac{\|U\|_\infty}{2c} \log \frac{1+t}{1+t-s},$$

by putting $k(t) = t/2$, we can see that

$$e^{\ell_t(k(t))} = \left(\frac{1+t}{1+t/2} \right)^{\|U\|_\infty/2c} \leq k_1$$

for some constant k_1. Hence, if F is a subset of $\{x : U(x) > b\}$ with $b > c$, then

$$\int_0^\infty e^{\ell_t(k(t))} \mu_t(F)\, dt \leq k_1 m(F) \int_0^\infty (1+t)^{-b/c} dt < \infty.$$

On the other hand, under the condition (44), taking ε_1 such that $\delta_1 \equiv 1 - (m_U/2c + \delta + \varepsilon_1) > 0$,

$$\begin{aligned}
\int_{t/2}^t \frac{1-\gamma(\tau)}{\gamma(\tau)} d\tau &\geq k_0 k_1(\varepsilon_1) \int_{t/2}^t (1+\tau)^{-(m_U/2c+\delta+\varepsilon_1)} d\tau \\
&= k_2(\varepsilon_1)(1+t)^{\delta_1} \left\{ 1 - \left(\frac{1+t/2}{1+t} \right)^{\delta_1} \right\} \\
&\geq k_2(\varepsilon_1) \left(1 - (2/3)^{\delta_1} \right) (1+t)^{\delta_1}
\end{aligned}$$

for $t \geq 2$, where $k_2(\varepsilon_1) = k_0 k_1(\varepsilon_1)/\delta_1$. Note that for any $\varepsilon_2 > 0$, we can find $k_3(\varepsilon_2)$ such that $\|I_F\|_{L^2(\mu_t)} \leq k_3(\varepsilon_2)(1+t)^{-b/2c-\varepsilon_2}$. So, we see

$$\begin{aligned}
&\int_2^\infty \|I_F\|_{L^2(\mu_t)} e^{-\frac{1}{2}\int_{t/2}^t S_{\gamma,\lambda}(\tau)\, d\tau} dt \\
&\leq \int_2^\infty \|I_F\|_{L^2(\mu_t)} e^{-\frac{1}{2}k(\gamma)^{-1}\int_{t/2}^t ((1-\gamma(\tau))/\gamma(\tau))\, d\tau} dt \\
&\leq k_3(\epsilon_2) \int_2^\infty (1+t)^{-b/2c-\epsilon_2} e^{-\frac{1}{2}k(\gamma)^{-1} k_2(\varepsilon_1)\left(1-(2/3)^{\delta_1}\right)(1+t)^{\delta_1}} dt < \infty.
\end{aligned}$$

Therefore, from Theorem 3.2, we have

$$E^\rho_{(0,x)} \left(\int_0^\infty I_F(X_t) dt \right) < \infty, \tag{45}$$

if $F \subset \{x : U(x) > b\}$ for $b > c$. Also, from Corollary 3.1,

$$\lim_{T\to\infty} P^\rho_{(0,x)} (X_t \in F \ for\ some\ t \geq T) = 0 \tag{46}$$

for such set F.

As a special case, consider the 1-dimensional torus $X = R^1/N$. In this case, since $\alpha(t) \leq (1/4)\|\Delta U\|_\infty \beta(t) + (1/8)\|\nabla U\|_\infty^2 \beta^2(t)$ (see §1.4 in [7]),

$$\Gamma(t) \geq \frac{k_1}{\alpha_U(t)} e^{-d\alpha_U(t)} \geq k_2(1+t)^{-d\|\nabla U\|_\infty/2c}$$

for the diameter $d = 1/2$ and constants k_1 and k_2, where

$$\alpha_U(t) = \sqrt{\frac{1}{2}\|\nabla U\|_\infty + \frac{1}{4}\beta^2(t)\|\nabla U\|_\infty^2}.$$

Hence (44) holds if $m_U + d\|\nabla U\|_\infty < 2c$. In the case of the present example, the spectral gap λ_1 is given in [6]. By using the spectral gap, more optimal condition is given there. In fact, it is shown that $m_U < c < b$ is enough to get (45) and (46) for $F \subset \{x : U(x) > b\}$.

Acknowledgement. The authors thanks to the referee for the valuable comments by which we could improve the estimate of Lemma 2.1.

References

[1] Azema, J., Kaplan-Duflo, M., Revuz, D. : Mesure invariante sur les récurrentes des processus de Markov, Z. Wahrscheinlichkeitstheorie verw. Geb. **8** (1967), 157–181.

[2] Blumenthal, R. M. and Getoor, R. K. : *Markov processes and Potential Theory*, Academic Press, 1968.

[3] Fitzsimmons, P. J. : On the quasi-regularity of semi-Dirichlet forms, Potential Anal. **15** (2001), 151–185.

[4] Fukushima, M., Oshima, Y. and Takeda, M. : *Dirichlet Forms and Symmetric Markov Processes*, Walter de Gruyter, Berlin-New York, 1994.

[5] Holley, R. A. and Stroock, D. W. : Simulated annealing via Sobolev inequalities, Comm. Math. Phys. **115** (1988), 553–569.

[6] Holley, R. A., Kusuoka, S. and Stroock, D. W. : Asymptotics of the spectral gap with applications to simulated annealing, J. Funct. Anal. **83** (1989), 333–347.

[7] Ito K. and McKean H.P. : *Diffusion Processes and their sample paths*, Springer Verlag, 1965.

[8] Meyn, S. P. and Tweedie, R. L. : Generalized resolvents and Harris recurrence of Markov processes, Contemp. Math. **149** (1993), 227–250.

[9] Oshima, Y. : On a construction of a recurrent potential kernel by means of time change and killing, J. Math. Soc. Japan, **29** (1977), 151–159

[10] Oshima, Y. : On the recurrence of some time inhomogeneous Markov processes, Forum Math. **10**, No 1. (1998), 119–134.

[11] Revuz, D. : Sur la théorie du potentiel pour les processus de Markov récurrents, Ann. Inst. Fourier (Grenoble), **21** (1971), 245–262

[12] Stollmann, P. and Voigt, J. : Perturbation of Dirichlet forms by measures, Potential Anal. **5** (1996), 109–138.

Daehong Kim
Department of mathematics and engineering,
Faculty of engineering, Kumamoto University,
Kumamoto 860-8555, Japan.

Yoichi Oshima
Department of mathematics and engineering,
Faculty of engineering, Kumamoto University,
Kumamoto 860-8555, Japan.