On a construction of space-time diffusion processes with boundary condition

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1 Introduction

Let D be a bounded domain of $\mathbb{R}^d$ with $\mathcal{C}^2$-class boundary ∂D. Throughout in this paper, we assume that we are given a family of bounded functions $(a_{ij}(\tau, x))$ on $[0.\infty) \times D$ which is symmetric and uniformly elliptic relative to x, that is $a_{ij}(\tau, x) = a_{ji}(\tau, x)$ and there exist positive conatants $\lambda_1 \leq \lambda_2$ such that

$$\lambda_1 \sum_{i=1}^{d} p_i^2 \leq \sum_{i,j=1}^{d} a_{ij}(\tau, x) p_i p_j \leq \lambda_2 \sum_{i=1}^{d} p_i^2 \tag{1.1}$$

for all $(\tau, x) \in [0, \infty) \times D$ and $\mathbf{p} = (p_1, p_2, \ldots, p_d) \in \mathbb{R}^d$.

Let $\{b_i(\tau, x)\}_{1 \leq i \leq d}$ and $\{\beta_i(\tau, x)\}_{1 \leq i \leq d}$ be families of bounded measurable functions on $[0, \infty) \times D$ and $[0, \infty) \times \partial D$, respectively. Furthermore assume that $\sum_{i=1}^{d} \beta_i(\tau, x) \frac{\partial}{\partial x_i}$ is a vector field on the tangent space of ∂D and the derivative of $\beta_i(\tau, x)$ along ∂D is uniformly bounded relative to $(\tau, x) \in [0, \infty) \times \partial D$. We consider that $\{a_{ij}(\tau, x)\}$, $\{b_i(\tau, x)\}$ and $\{\beta_i(\tau, x)\}$ are defined for $\tau < 0$ by putting $a_{ij}(\tau, x) = a_{ij}(0, x)$, $b_i(\tau, x) = b_i(0, x)$ and $\beta_i(\tau, x) = \beta_i(0, x)$ for $\tau < 0$.

Since D is of compact closure, its boundary ∂D is covered by a finite number of open sets $\{O^{(k)}\}_{1 \leq k \leq n}$. Furthermore, for each $k \geq 1$, there exists an open set $U^{(k)}$ of $\mathbb{R}^d$ and a family of $\mathcal{C}^2$-functions $\Phi^{(k)}(\xi)$ of $\xi = (\xi_1, \xi_2, \ldots, \xi_d)$ given by $\Phi^{(k)}(\xi) = \left(\varphi_1^{(k)}(\xi), \varphi_2^{(k)}(\xi), \cdots, \varphi_d^{(k)}(\xi)\right)$ such that any points of $U^{(k)}$ are mapped onto $O^{(k)}$ as one to one map and

$$\partial D \cap O^{(k)} = \{x = \Phi^{(k)}(\xi', 0) : (\xi', 0) \in U^{(k)}\},$$

where $\xi' = (\xi_1, \xi_2, \ldots, \xi_{d-1})$. Let $\{\delta^{(k)}(x)\}_{1 \leq k \leq n}$ be a partition of unity subordinated to $\{O^{(k)}\}$, that is $\delta^{(k)}$ is a $\mathcal{C}^\infty$ function supported by $O^{(k)}$ and $\sum_{k=1}^{n} \delta^{(k)}(x) = 1$ on a neighbourhood of ∂D. Let $\mathcal{C}^1(\overline{D})$ be the family of functions φ on $\overline{D}$ which can be extended to a function of $\mathcal{C}^1(\mathbb{R}^d)$ and F

be the Sobolev space of order 1, that is $F = \{\varphi \in L^2(\overline{D};m) : \partial\varphi/\partial x_i \in L^2(\overline{D};m)$ for all $1 \le i \le d\}$ for $m(dx) = dx$. Then F is equal to the closure of $\mathcal{C}^1(\overline{D})$ relative to the Dirichlet norm $\mathbf{D}_1$ (see [4] § 1.5), where $\mathbf{D}_1(\varphi,\psi) = \mathbf{D}(\varphi,\psi) + (\varphi,\psi)_H$ with $H = L^2(\overline{D}, m)$ and

$$\mathbf{D}(\varphi,\psi) = \int_D \sum_{i=1}^d \frac{\partial\varphi}{\partial x_i}\,\frac{\partial\psi}{\partial x_j}\,dx. \tag{1.2}$$

Define a bilinear form $E^{(\tau)}$ on F by

$$E^{(\tau)}(\varphi,\psi) = E^{(\tau),1}(\varphi,\psi) - E^{(\tau),2}(\varphi,\psi) \tag{1.3}$$

for

$$\begin{aligned} E^{(\tau),1}(\varphi,\psi) &= \sum_{i,j=1}^d \int_D a_{ij}(\tau,x)\,\frac{\partial\varphi}{\partial x_i}\,\frac{\partial\psi}{\partial x_j}dx \\ &\quad -\sum_{i=1}^d \int_D b_i(\tau,x)\,\frac{\partial\varphi}{\partial x_i}\,\psi(x)dx \qquad (1.4) \\ E^{(\tau),2}(\varphi,\psi) &= \sum_{i=1}^d \int_{\partial D} \beta_i(\tau,x)\,\frac{\partial f}{\partial x_i}\,g(x)dS, \qquad (1.5) \end{aligned}$$

where dS is the surface element of ∂D.

The purpose of this paper is to construct a space-time diffusion process correspoding to the time dependent family of semi-Dirichlet forms $(E^{(\tau)}, F)$ with $\tau \ge 0$. The corresponding diffusion is associated with a parabolic differential generator with oblique reflecting boundary condition (see Fukushima [2], Kim [5] and Tsuchiya [12]). Our construction is based upon the general theory of time dependent semi-Sirichlet forms given by Ma, Overbeck and Röckner [6], Ma and Röckner [7] and Oshima [11].

For a subspace F of H and a bilinear form $E(\varphi,\psi)$ defined for $\varphi,\psi \in F$, (E, F) is called a semi-Dirichlet form on H if it satisfies the following conditions:

(E.1) (Lower boundedness): There exists a positive constant α_0 such that $E_{\alpha_0}(\varphi,\varphi) \ge 0$ for all $\varphi \in F$.

(E.2) (Sector condition): There exists a constant K such that

$$|E(\varphi,\psi)| \le K E_{\alpha_0}(\varphi,\varphi)^{1/2} E_{\alpha_0}(\psi,\psi)^{1/2} \tag{1.6}$$

for all $\varphi,\psi \in F$.

$(E.3)$ (Closedness): F is a Hilbert space relative to the symmetric bilinear form $E^s_\alpha(\varphi,\psi) = \frac{1}{2}\left(E_\alpha(\varphi,\psi) + E_\alpha(\psi,\varphi)\right)$ for any $\alpha > \alpha_0$.

$(E.4)$ (Markov property): For any $\varphi \in F$ and non-negative constant a, $\varphi \wedge a \in F$ and $E(\varphi \wedge a, \varphi \wedge a) \le E(\varphi \wedge a, \varphi)$.

In Section 2, we shall show that $(E^{(\tau)}, F)$ is a semi-Dirichlet form on H for any τ with common constants α_0 and K. Under our present assumption, we shall show that the symmetric Dirichlet form $(\mathbf{D}, H^1(\overline{D})$ with the Dirichlet integral

$$\mathbf{D}(u,v) = \int_D \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx$$

is a reference form of $(E^{(\tau)}, F)$, that is, for any $\alpha > \alpha_0$, there exist positive constants $c_1 \le c_2$ such that

$$c_1\,\mathbf{D}_1(u,u) \le \mathcal{E}_\alpha(u,u) \le c_2\mathbf{D}_1(u,u)$$

for all $u \in F$. In particular, F satisfies the condition $(\mathcal{E}.5)$ given in [11] which essentially implies that $uv \in F_b$ for any $u, v \in F_b$, where F_b is the family of bounded functions of F. For any τ and $\delta > 0$, take a strictly positive δ-coexcessive function $\widehat{h}_\delta(\tau,\cdot) \in F$. As in [11] Theorem 2.4.8, we may assume that $\widehat{h}_\delta(x,\cdot) \ge c_\delta(\tau) > 0$ for some positive constant $c_\delta(\tau)$ which is uniformly bounded from below by a positive constant on each finite τ-interval. We shall also show that $(E^{(\tau)}, F)$ satisfies condition $(E.6)$ which is a weaker version of the sector condition of the bilinear form $A^{(\tau),\delta}(\varphi,\psi) = E^{(\tau)}(\varphi, \psi\widehat{h}_\delta(\tau,\cdot))$.

In Section 3, we construct the associated space-time process and give a probabilistic representation of a weak solution of a parabolic differential equation with boundary condition. Put $Z = R^1 \times \overline{D}$ and let ν be the measure on Z given by $d\nu(\tau,x) = d\tau \times dm(x)$. Let $\mathcal{H} = L^2(\mathbb{R}^1; H)$ be the family of functions $u(\tau,x)$ such that $u(\tau,\cdot) \in H$ for all $\tau \in \mathbb{R}^1$ and

$$\|u\|^2_{\mathcal{H}} = \int_{\mathbb{R}^1} \|u(\tau,\cdot)\|^2_H\,d\tau < \infty.$$

Define $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ similarly by using F instead of H in the definition of $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$. By considering that $F \subset H = H'$, let F' be the dual space of F and define $\mathcal{F}' = L^2(\mathbb{R}^1; F')$ similarly. For $u \in \mathcal{F}$, let $\frac{\partial u}{\partial \tau}$ be the distribition sense derivative of $\tau \mapsto u(\tau,\cdot) \in F'$, that is $\int_{\mathbb{R}^1} \partial u(\tau,\cdot)/\partial\tau\xi(\tau)\,d\tau = \int_{bR^1} u(\tau,\cdot)\xi'(\tau)\,d\tau$ as an element of F' for any $\xi \in \mathcal{C}^\infty_0(\mathbb{R}^1)$. Define a family $\mathcal{W}$ by $\mathcal{W} = \{u \in \mathcal{F} : \|u\|_{\mathcal{W}} < \infty\}$ for $\|u\|^2_{\mathcal{W}} = \|\partial u/\partial\tau\|^2_{\mathcal{F}'} + \|u\|^2_{\mathcal{F}}$ and let

$$\mathcal{E}(u,v) = \begin{cases} -\left(\frac{\partial u}{\partial \tau}, v\right) + \mathcal{B}(u,v) & u \in \mathcal{W}, v \in \mathcal{F} \\ \left(\frac{\partial v}{\partial \tau}, u\right) + \mathcal{B}(u,v) & u \in \mathcal{F}, v \in \mathcal{W}, \end{cases} \tag{1.7}$$

where

$$\left(\frac{\partial u}{\partial \tau}, v\right) = \int_{bR^1} {}_{F'}\left(\frac{\partial u}{\partial \tau}, v(\tau,\cdot)\right)_F d\tau,$$
$$\mathcal{B}(u,v) = \int_{\mathbb{R}^1} E^{(\tau)}(u(\tau,\cdot), v(\tau,\cdot))\, d\tau.$$

We say that $(\mathcal{E},\mathcal{F})$ a *time dependent semi-Dirichlet form.* Then there exists a resolvent $\{G_\alpha\}_{\alpha>0}$ such that $G_\alpha f \in \mathcal{W}$ for any $f \in \mathcal{H}$ and

$$\mathcal{E}_\alpha(G_\alpha f, v) = (f,v) \quad \text{for any } f \in \mathcal{F}'.$$

By choosing a resolvent kernel $R_\alpha(x,dy)$ such that $R_\alpha f(x) = \int_{\overline{D}} R_\alpha(x,dy) f(f)$ is a q.c. modification of $G_\alpha f$ for any $f \in \mathcal{H}$ and $\alpha > \alpha_0$, there corresponds a diffusion process $Z_t = (\tau(t), X_t)$ on Z with resolvent $\{R_\alpha\}$. Then $\tau(t)$ is the uniform motion to the right, that is $\tau(t) = \tau(0) + t$. By the Sobolev inequality, we can see that the resolvent $R_\alpha(x,dy)$ can be chosen as $R_\alpha(x,\cdot)$ is absolutely continuous relative to m (see Davies [1] and Fukushima [3]).

We will show in Section 3 that the measure $d\tau \times dS(x)$ is a smooth measure of the space-time process. By using the local time associated with the smooth measure $d\tau \times dS$, we gve a probabilistic representation of the weak solution of

$$\frac{\partial u}{\partial \tau} = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(\tau,x)\frac{\partial u}{\partial u}\right) + \sum_{i=1}^{d} b_i(\tau,x)\frac{\partial u}{\partial x_i} \tag{1.8}$$

on D with boundary condition

$$\frac{\partial u}{\partial \mathbf{n}} + \sum_{i=1}^{d} \beta_i(\tau,x)\frac{\partial u}{x_i} = g(\tau,x) \tag{1.9}$$

on ∂D and terminal condition $u(T,x) = \varphi(x)$ for $x \in D$. Such representation is given by M. Tsuchiya [13] for more general space-time domain and boundary condition.

For a fixed $\delta > \alpha_0$, take a positive quasi-continuous δ-coexcessive function $\widehat{h}_\delta \in \mathcal{F}$. We may assume that $\widehat{h}_\delta(\tau,x)$ is bounded from below by a positive constant on $[a,b]\times\overline{D}$ for any finite interval $[a,b]$ (see [11] Theorem 2.4.8). Define the energy $e^{(\delta)}(A)$ of an additive functional A_t by

$$e^{(\delta)}(A) = \frac{1}{2}\lim_{t\to 0}\frac{1}{t}\mathbb{E}_{\widehat{\nu}}(A_t^2), \tag{1.10}$$

where $\widehat{\nu}(\tau, x) = \widehat{h}_\delta(\tau, x) d\nu$. Let $fien$ be the family of martingale additive functionals M such that $e^{(\delta)}(M) < \infty$ and $zeen$ be the family of continuous additive functionals N such that $e^{(\delta)}(N) = 0$, respectively. In [11], only the weak sense Fukushima's decomposition is given explicitly in the time dependent case. Under our present setting, the Fukushima's decomposition relative to the energy $e^{(\delta)}$ is possible. In Section 3, we also mention about the decomposition and give the characterizations of the martingale and zero energy parts. See also [8] for such decomposition.

2 Semi-Dirichlet form property of $E^{(\tau)}$

Let $(\mathbf{D}, F)$ be the Dirichlet form on H defined by (1.2). For a bilinear form (E, F), $\mathbf{D}$ is called a reference form of E if there exists a non-negative constant α_0 such that, for any $\alpha > \alpha_0$ we can choose a suitable constants $c_1(\alpha)$ and $c_2(\alpha)$ satisfying

$$c_1(\alpha)\mathbf{D}_1(\varphi, \varphi) \leq E_\alpha(\varphi, \varphi) \leq c_2(\alpha)\mathbf{D}_1(\varphi, \varphi) \tag{2.1}$$

for all $\varphi \in F$. Clearly, (2.1) implies that $E_{\alpha_0}(\varphi, \varphi) \geq 0$.

Note that $\mathbf{D}$ is a reference form of E if (2.1) holds for some $\alpha_1 > \alpha_0$. In fact, if (2.1) holds for some $\alpha_1 > \alpha_0$, then for any $\alpha \geq \alpha_1$, $E_\alpha(\varphi, \varphi) \geq E_{\alpha_1}(\varphi, \varphi) \geq c_1(\alpha_1)\mathbf{D}(\varphi, \varphi)$. On the other hand, if $\alpha_1 > \alpha > \alpha_0$, then

$$\begin{aligned} E_\alpha(\varphi, \varphi) &= E_{\alpha_0}(\varphi, \varphi) + (\alpha - \alpha_0)(\varphi, \varphi) \\ &\geq \frac{\alpha - \alpha_0}{\alpha_1 - \alpha_0} \left(E_{\alpha_0}(\varphi, \varphi) + (\alpha_1 - \alpha_0)(\varphi, \varphi)\right) \\ &\geq c_1(\alpha)\mathbf{D}_1(\varphi, \varphi) \end{aligned}$$

for $k_1(\alpha) = c_1(\alpha_1)(\alpha - \alpha_0)/(\alpha_1 - \alpha_0)$. Therefore, for any $\alpha > \alpha_0$, by putting $c_1(\alpha) = k_1(\alpha_1) \times ((\alpha - \alpha_0)/(\alpha_1 - \alpha_0)) \wedge 1$, it holds that $c_1(\alpha)\mathbf{D}_1(\varphi, \varphi) \leq E_\alpha(\varphi, \varphi)$. Similarly, the second inequality $E_\alpha(\varphi, \varphi) \leq c_2(\alpha)\mathbf{D}_1(\varphi, \varphi)$ of (2.1) holds for $c_2(\alpha) = c_2(\alpha_1) \times ((\alpha - \alpha_0)/(\alpha_1 - \alpha_0)) \vee 1$.

Lemma 2.1. $(\mathbf{D}, F)$ *is a reference form of* $(E^{(\tau),1}, F)$ *for all* τ*, that is there exists a constant* $\alpha_0 \geq 0$ *such that, for any* $\alpha > \alpha_0$*, we can find positive constants* c_3, c_4 *satisfying*

$$c_3\mathbf{D}_1(\varphi, \varphi) \leq E_\alpha^{(\tau),1}(\varphi, \varphi) \leq c_4\mathbf{D}_1(\varphi, \varphi) \tag{2.2}$$

for all $\varphi \in F$*.*

Proof. Put $k = \sup\{\lambda(\tau) : \tau \geq 0\}$. Then, for any $\tau \geq 0$,

$$\begin{aligned} E^{(\tau),1}(\varphi,\varphi) &= \int_D \sum_{i,j=1}^d a_{ij}(\tau,x)\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\,dx \\ &\quad - \int_D b_i(\tau,x)\frac{\partial\varphi}{\partial x_i}\,\psi(x)dx \\ &\geq \lambda_1\mathbf{D}(\varphi,\varphi) - \|b\|_\infty\mathbf{D}(\varphi,\varphi)^{1/2}\|\varphi\| \\ &\geq \lambda_1\mathbf{D}(\varphi,\varphi) - \frac{1}{2}\left(\lambda_1\mathbf{D}(\varphi,\varphi) + \frac{1}{\lambda_1}\|b\|_\infty^2\|\varphi\|^2\right). \end{aligned}$$

Hence,

$$\frac{\lambda_1}{2}\mathbf{D}(\varphi,\varphi) \leq E_{\beta_1}^{(\tau),1}(\varphi,\varphi)$$

for $\beta_1 = \frac{1}{2\lambda_1}\|b\|_\infty^2$. Hence

$$c_3\mathbf{D}_1(\varphi,\varphi) \leq E_{\beta_2}^{(\tau),1}(\varphi,\varphi)$$

for $c_3 = \lambda_1/2$ and $\beta_2 = \beta_1 + 1/2\lambda_1$.

Conversely, for any $\varphi,\psi \in F$,

$$\begin{aligned} |E^{(\tau),1}(\varphi,\psi)| &\leq \lambda_2\mathbf{D}(\varphi,\varphi)^{1/2}\mathbf{D}(\psi,\psi)^{1/2} + \|b\|_\infty\mathbf{D}(\varphi,\varphi)^{1/2}\|\psi\| \\ &\leq \lambda_2\mathbf{D}(\varphi,\varphi)^{1/2}\left(\mathbf{D}(\psi,\psi)^{1/2} + \|b\|_\infty\|\psi\|\right). \end{aligned}$$

Hence, noting that $\sqrt{a}+\sqrt{b} \leq \sqrt{2(a+b)}$, we have

$$\begin{aligned} |E_{\beta_2}^{(\tau),1}(\varphi,\psi)| &\leq \sqrt{2}\lambda_2\mathbf{D}(\varphi,\varphi)^{1/2}\mathbf{D}_{\|b\|_\infty}(\psi,\psi)^{1/2} + \beta_2\|\varphi\|\,\|\psi\| \\ &\leq (\sqrt{2}\lambda_2 + \beta_2 + \|b\|_\infty)\mathbf{D}_1(\varphi,\varphi)^{1/2}\mathbf{D}_1(\psi,\psi)^{1/2}. \end{aligned}$$

Hence

$$|E_{\beta_2}^{(\tau),1}(\varphi,\psi)| \leq c_4\mathbf{D}_1(\varphi,\varphi)^{1/2}\mathbf{D}_1(\psi,\psi)^{1/2}$$

for $c_4 = \sqrt{2}\lambda_2 + \beta_2 + \|b\|_\infty$. As we noted before the statement of the lemma, this implies the assertion. □

Let $\{O^{(k)}\}_{1\leq k\leq n}$ be a covering of ∂D and $\{\delta^{(k)}(x)\}$ a partition of unity subordinated to $\{O^{(k)}\}$ defined in Section 1. Let $\varphi \in \mathcal{C}^1(\overline{D})$. Then we may assume that φ is a restriction of a function of $\mathcal{C}^1(\mathbb{R}^d)$, which is also denoted by φ, on D. Decompose φ as

$$\varphi(x) = \sum_{k=1}^n \varphi(x)\delta^{(k)}(x) + \varphi(x)\left(1 - \sum_{k=1}^n \delta^{(k)}(x)\right) \equiv \varphi_1(x) + \varphi_2(x). \quad (2.3)$$

Then $\varphi_2 \in \mathcal{C}_0^1(D)$ and hence $E^{(\tau)}(\varphi_2, \psi) = E^{(\tau),1}(\varphi_2, \psi)$ for any $\psi \in \mathcal{C}^1(\overline{D})$.

Since

$$E^{(\tau),2}(\varphi, \psi) = \sum_{k=1}^{d} E^{(\tau),2}(\varphi\delta^{(k)}, \psi),$$

by considering each term $E^{(\tau),2}(\varphi\delta_i, \psi)$ separately, we may assume that φ is supported by $O = O^{(k)}$ for fixed k. Put $\Phi = \Phi^{(k)}$, then by the mapping $\xi = \Phi(x)$ given in § 1.1, O is mapped onto $V \subset \mathbb{R}^d$ and $D \cap O$ is mapped onto $V \cap \{\xi = (\xi_1, \dots, \xi_{d-1}, \xi_d) \in \mathbb{R}^d : \xi_d > 0\}$. Let $x = \Psi(\xi) = (\psi_1(\xi), \psi_2(\xi), \dots, \psi_d(\xi))$ be the inverse map of Φ and put $\Gamma = (\gamma_{ij})$ for $\gamma_{ij} = \partial\psi_i/\partial\xi_j$. Then $dx = \rho(\xi)d\xi$ for $\rho(\xi) = \det(\Gamma)$. Furthermore the surface measure is given by $dS = \overline{S}(\xi', 0))d\xi'$, where $\xi' = (\xi_1, \xi_2, \dots, \xi_{d-1})$ and $\overline{S}(\xi) = \left(\sum_{i=1}^d \Gamma_{id}^2(\xi)\right)^{1/2}$ by using the cofactor Γ_{id} of γ_{id} of Γ.

For any φ and $\psi \in \mathcal{C}^1(\overline{D})$, put $\overline{\varphi}(\xi) = \varphi(\Psi(\xi))$ and $\overline{\psi}(\xi) = \psi(\Psi(\xi))$, respectively. Since $\sum_{i=1}^d \beta_i(x)\partial\varphi/\partial x_i$ for $x \in \partial D$ is a derivative along the tangent space of ∂D, it is transformed by the mapping $x = \Psi(\xi)$ to a derivative on $\mathbb{R}^{d-1} \times \{0\}$, that is to

$$\sum_{j=1}^{d-1} \sum_{i=1}^{d} \beta_i(\Phi(\xi', 0))\gamma_{ij}^{-1}(\xi', 0)\, \frac{\partial\overline{\varphi}(\xi', 0)}{\partial\xi_j},$$

where (γ_{ij}^{-1}) is the inverse matrix of Γ. Therefore, by putting $\overline{\beta}_j(\xi') = \sum_{i=1}^d \beta_i(\Phi(\xi', 0))\gamma_{ij}^{-1}(\xi', 0)$ we can write $E^{(\tau),2}(\varphi, \psi)$ as

$$\begin{aligned} E^{(\tau),2}(\varphi, \psi) &= \int_{\partial D} \sum_{i=1}^{d} \beta_i(x)\frac{\partial\varphi}{\partial x_i}\psi(x)dS(x) \\ &= \int_{\mathbb{R}^{d-1}} \sum_{j=1}^{d-1} \overline{\beta}_j(\xi')\frac{\partial\overline{\varphi}(\xi', 0)}{\partial\xi_j}\overline{\psi}(\xi', 0)\overline{S}(\xi', 0)d\xi'. \end{aligned} \tag{2.4}$$

Lemma 2.2. *There exist positive constants c_5, c_6 satisfying*

$$\left|E^{(\tau),2}(\varphi, \psi)\right| \le c_5\mathbf{D}_1(\varphi, \varphi)^{1/2}\mathbf{D}_1(\psi, \psi)^{1/2} \tag{2.5}$$

$$E^{(\tau),2}(\varphi, \varphi) \ge -\epsilon\mathbf{D}(f, f) - \frac{c_6}{\epsilon}(\varphi, \varphi) \tag{2.6}$$

for all $\varphi, \psi \in F$ and $\epsilon > 0$.

Proof. Assume first that φ is supported by $O = O^{(k)}$ for sme k. Further, approximating φ by functions of $\mathcal{C}_0^2(\overline{D}\cap O)$, we may assume that $\varphi \in \mathcal{C}_0^2(\overline{D}\cap O)$ and hence we may consider that φ is a restriction of a function $\varphi \in \mathcal{C}_0^2(O)$ to $\overline{D}$. The corresponding function $\overline{\varphi}$ by the mapping Φ is also considered as a restriction of $\overline{\varphi} \in \mathcal{C}_0^2(V)$ to $\mathbb{R}_+^d = \{\xi = (\xi_1, \xi_2, \ldots, \xi_d) : \xi_d \geq 0\}$. Furthermore, we may assume that $\overline{\varphi} \in \mathcal{C}_0^2(\mathbb{R}^d)$ by putting $\overline{\varphi} = 0$ outside of V. By (2.4), we can write

$$
\begin{aligned}
-E^{(\tau),2}(\varphi,\psi) &= \sum_{j=1}^{d-1} \int_{\mathbb{R}^{d-1}} \int_0^\infty \frac{\partial}{\partial \xi_d} \left(\overline{\beta}_j(\xi') \frac{\partial \overline{\varphi}}{\partial \xi_j} \overline{\psi}(\xi', \xi_d) \overline{S}(\xi', \xi_d) \right) d\xi' d\xi_d \\
&= \sum_{j=1}^{d-1} \int_{bR^{d-1}} \int_0^\infty \overline{\beta}(\xi') \frac{\partial^2 \overline{\varphi}}{\partial \xi_j \partial \xi_d} \overline{\psi}(\xi) \overline{S}(\xi) d\xi \\
&\quad + \sum_{j=1}^{d-1} \int_{\mathbb{R}^{d-1}} \overline{\beta}(\xi') \frac{\partial \overline{\varphi}}{\partial \xi_j} \frac{\partial \overline{\psi}}{\partial \xi_d} \overline{S}(\xi) d\xi \\
&\quad + \sum_{j=1}^{d-1} \int_{\mathbb{R}^{d-1}} \overline{\beta}(\xi') \frac{\partial \overline{\varphi}}{\partial \xi_j} \overline{\psi}(\xi) \frac{\partial \overline{S}}{\partial \xi_d} d\xi \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}. \qquad (2.7)
\end{aligned}
$$

For the inward unit normal vector $\mathbf{n}$ of ∂D and the unit vector $\mathbf{d}$ in the direction of $(\gamma_{id})_{1\leq i \leq d}$, since $\rho(\xi) = \overline{S}(\xi) \left(\sum_{i=1}^d \gamma_{id}^2 \right)^{1/2} (\mathbf{n}, \mathbf{d})$ is positive and continuous, there exist positive constants k_1 and k_2 satisfying $k_1 \overline{S}(\xi) \leq \rho(\xi) \leq k_2 \overline{S}(\xi)$ for all $\xi \in V \cap \mathbb{R}_+^d$. Hence, by putting $\|\overline{\beta}\|_\infty = \max\{\|\overline{\beta}_i\|_\infty : 1 \leq i \leq d\}$, it holds that

$$
\begin{aligned}
\mathrm{II} &\leq \|\overline{\beta}\|_\infty \sum_{j=1}^{d-1} \left(\int_{\mathbb{R}_+^d} \left(\frac{\partial \overline{\varphi}}{\partial \xi_j} \right)^2 \overline{S}(\xi) d\xi \right)^{1/2} \left(\int_{\mathbb{R}_+^d \cap V} \left(\frac{\partial \overline{\psi}}{\partial \xi_d} \right)^2 \overline{S}(\xi) d\xi \right)^{1/2} \\
&\leq \frac{1}{k_1} \|\overline{\beta}\|_\infty \mathbf{D}(\varphi,\varphi)^{1/2} \mathbf{D}(\psi,\psi)^{1/2}.
\end{aligned}
$$

Since $\frac{1}{\sqrt{\overline{S}(\xi)}} \frac{\partial \overline{S}}{\partial \xi_d}$ is bounded by some constant k_3 on $V \cap \mathbb{R}_+^d$,

$$
\begin{aligned}
\mathrm{III} &\leq k_3 \|\overline{\beta}\|_\infty \sum_{j=1}^{d-1} \left(\int_{\mathbb{R}_+^d \cap V} \left(\frac{\partial \overline{\varphi}}{\partial \xi_j} \right)^2 \overline{S}(\xi) d\xi \right)^{1/2} \left(\int_{\mathbb{R}_+^d \cap V} \overline{\psi}^2(\xi) \overline{S}(\xi) d\xi \right)^{1/2} \\
&\leq \frac{k_3}{k_1} \|\overline{\beta}\|_\infty \mathbf{D}(\varphi,\varphi)^{1/2} \mathbf{D}_1(\psi,\psi)^{1/2}.
\end{aligned}
$$

For fixed ξ_d, since $\overline{\varphi}(\xi', \xi_d)$ has compact support relative to ξ', by the integration by parts formula, I can be written as

$$
\begin{aligned}
\mathrm{I} &= -\sum_{j=1}^{d-1} \int_{\mathbb{R}^d_+ \cap V} \frac{\partial \overline{\varphi}}{\partial \xi_d} \frac{\partial \overline{\psi}}{\partial \xi_j} \overline{\beta}_j(\xi') \overline{S}(\xi) d\xi \\
&\quad - \sum_{j=1}^{d-1} \int_{\mathbb{R}^d_+ \cap V} \frac{\partial \overline{\varphi}}{\partial \xi_d} \overline{\psi}(\xi) \frac{\partial}{\partial \xi_j} \left(\overline{\beta}_j(\xi') \overline{S}(\xi) \right) d\xi.
\end{aligned}
$$

We may assume that k_3 also satisfies $(1/\overline{S}(\xi))|\partial \overline{S}/\partial \xi_j| < k_3$ on $\mathbb{R}^d_+ \cap V$ for all $1 \leq j \leq d-1$. Furthermore, there exists a constant k_4 such that $\|\partial \overline{\beta}_j / \partial \xi_j\|_\infty \leq k_4$ for all $j \leq d-1$. Hence, similarly to II and III, it holds that

$$
\begin{aligned}
|\mathrm{I}| &\leq \frac{1}{k_1} \|\overline{\beta}\|_\infty \mathbf{D}(\varphi, \varphi)^{1/2} \mathbf{D}(\psi, \psi)^{1/2} \\
&\quad + \frac{1}{k_1} (k_3 \|\overline{\beta}\|_\infty + k_4) \mathbf{D}(\varphi, \varphi)^{1/2} \mathbf{D}_1(\psi, \psi)^{1/2}.
\end{aligned}
$$

Therefore we obtained (2.5) for $\varphi \in \mathcal{C}^1(O \cap \overline{D})$.

For general $\varphi \in \mathcal{C}^1(\overline{D})$, put $\varphi_k = \varphi \delta^{(k)}$. Then, for each k, there exists a constant γ_k independent of φ such that (2.5) holds for φ_k instead of φ. Hence

$$
|E^{(\tau),2}(\varphi, \psi)| \leq \sum_{k=1}^n |E^{(\tau),2}(\varphi_k, \psi)| \leq \sum_{k=1}^n \gamma_k \mathbf{D}_1(\varphi_k, \varphi_k)^{1/2} \mathbf{D}_1(\psi, \psi)^{1/2}.
$$

Clearly $(\varphi_k, \varphi_k) \leq (\varphi, \varphi)$. Furthermore, since

$$
\begin{aligned}
\left(\frac{\partial \varphi_k}{\partial x_i} \right)^2 &\leq 2 \left(\left(\frac{\partial \varphi}{\partial x_i} \right)^2 + \varphi^2(x) \left\| \frac{\partial \delta^{(k)}}{\partial \xi_i} \right\|_\infty \right) \\
&\leq 2k_5 \left(\left(\frac{\partial \varphi}{\partial x_i} \right)^2 + \varphi^2(x) \right)
\end{aligned}
$$

for $k_5 = 1 \vee \max\{\|\partial \delta^{(k)} / \partial \xi_i\|_\infty : 1 \leq i \leq n\}$, we obtain that

$$
\begin{aligned}
|E^{(\tau),2}(\varphi, \psi)| &\leq 2k_5 \sum_{k=1}^n \gamma_k \left(\mathbf{D}(\varphi, \varphi) + (d+1)(\varphi, \varphi) \right)^{1/2} \mathbf{D}_1(\psi, \psi)^{1/2} \\
&\leq c_5 \mathbf{D}(\varphi, \varphi)^{1/2} \mathbf{D}_1(\psi, \psi)^{1/2}
\end{aligned}
$$

for $c_5 = 2k_5(d+1)^{1/2}\sum_{k=1}^n \gamma_k$.

To show (2.6), take an arbitrary positive ϵ. Similarly to (2.7),

$$
\begin{aligned}
E^{(\tau),2}(\varphi,\varphi) &= -\frac{1}{2}\sum_{j=1}^{d-1}\int_{\mathbb{R}^d_+}\frac{\partial}{\partial \xi_d}\left(\overline{\beta}_j(\xi')\,\frac{\partial \overline{\varphi}^2}{\partial \xi_j}\,\overline{S}(\xi)\right)d\xi \\
&= -\frac{1}{2}\sum_{j=1}^{d-1}\int_{\mathbb{R}^d_+}\overline{\beta}_j(\xi')\,\frac{\partial^2 \overline{\varphi}^2}{\partial \xi_j \partial \xi_d}\,\overline{S}(\xi) d\xi \\
&\quad -\frac{1}{2}\sum_{j=1}^{d-1}\int_{\mathbb{R}^d_+}\overline{\beta}_j(\xi')\,\frac{\partial \overline{\varphi}^2}{\partial x_j}\,\frac{\partial \overline{S}}{\partial \xi_d} d\xi \\
&= \sum_{j=1}^{d-1}\int_{\mathbb{R}^d_+}\frac{\partial \overline{\varphi}}{\partial \xi_j}\,\overline{\varphi}(\xi)\,\frac{\partial(\overline{\beta}_j\overline{S})}{\partial \xi_d} d\xi \\
&\quad -\int_{\mathbb{R}^d_+}\sum_{j=1}^{d-1}\overline{\beta}_j(\xi')\,\frac{\partial \overline{\varphi}}{\partial x_j}\,\overline{\varphi}(\xi)\,\frac{\partial \overline{S}}{\partial \xi_d} d\xi \\
&\geq -\frac{1}{k_1}(k_3\|\overline{\beta}\|_\infty + k_4 + k_3)\mathbf{D}(\varphi,\varphi)^{1/2}\|\varphi\| \\
&\geq -\epsilon \mathbf{D}(\varphi,\varphi) - \frac{c_6}{\epsilon}\,(\varphi,\varphi)
\end{aligned}
$$

for $c_6 = 4(1/k_1)^2(k_3\|\overline{\beta}\|_\infty + k_4 + k_3)^2$. This implies the assertion for $\varphi \in \mathcal{C}^1(O \cap \overline{D})$. Extension to any $\varphi \in \mathcal{C}^1(\overline{D})$ is similar to the proof of (2.5). $\square$

Theorem 2.3. *For any $\tau \geq 0$, $(E^{(\tau)}, F)$ is a semi-Dirichlet form on $L^2(D;dx)$ with reference form $(\mathbf{D}, F)$, that is there exists positive constant $c_1(\alpha)$ and $c_2(\alpha)$ satisfying (2.1) with $E^{(\tau)}_\alpha$ instead of E_α. Furthermore, the constants α_0, K in $(E.1), (E.2)$ and $c_1(\alpha)$, $c_2(\alpha)$ can be chosen independently of τ.*

Proof. By virtue of (2.2), there exist constants $\alpha_0 \geq 0$ and c_3 such that $\mathcal{E}^{(\tau),1}(\varphi,\varphi) \geq c_3\mathbf{D}_1(\varphi,\varphi)$. Take ϵ such that $\epsilon < c_3$, then by (2.6),

$$E^{(\tau)}(\varphi,\varphi) = E^{(\tau),1}(\varphi,\varphi) + E^{(\tau),2}(\varphi,\varphi) \geq c_3\mathbf{D}_1(\varphi,\varphi) - \epsilon\mathbf{D}(\varphi,\varphi) - \frac{c_6}{\epsilon}(\varphi,\varphi).$$

Hence

$$E^{(\tau)}_{\alpha_1}(\varphi,\varphi) = (c_1 - \epsilon)\mathbf{D}(\varphi,\varphi) \geq 0, \tag{2.8}$$

for $\alpha_1 = (c_6/\epsilon) - c_3) \vee 0$. This implies the lower boundedness of $(E^{(\tau)}, F)$.

For any fixed $\alpha_2 > \alpha_1$, since $(c_3 - \epsilon)\mathbf{D}_1(\varphi, \varphi) \leq E^{(\tau)}_{\alpha_0}(\varphi, \varphi)$ for $\alpha_0 = \alpha_1 + (c_3 - \epsilon)$, by using (2.2) and (2.5), it follws that

$$\begin{aligned} |E^{(\tau)}(\varphi, \psi)| &\leq c_4 \mathbf{D}_1(\varphi, \varphi)^{1/2} \mathbf{D}_1(\psi, \psi)^{1/2} + c_5 \mathbf{D}_1(\varphi, \varphi)^{1/2} \mathbf{D}_1(\psi, \psi)^{1/2} \\ &\leq K E^{(\tau)}_{\alpha_0}(\varphi, \varphi)^{1/2} E^{(\tau)}_{\alpha_0}(\psi, \psi)^{1/2} \end{aligned}$$

for $K = (c_4 + c_5)/(c_3 - \epsilon)$. Hence the sector condition holds.

In the above proof, we have seen that

$$\begin{aligned} (c_3 - \epsilon)\mathbf{D}_1(\varphi, \varphi) &\leq E^{(\tau)\alpha_0}(\varphi, \varphi) \leq (c_4 + c_5)\mathbf{D}_1(\varphi, \varphi) + \alpha_0(\varphi, \varphi) \\ &\leq (c_4 + c_5) \vee \alpha_0 \mathbf{D}_1(\varphi, \varphi). \end{aligned}$$

This implies (2.1) for $\alpha = \alpha_0$ by putting $c_1(\alpha_0) = c_3 - \epsilon$ and $c_2(\alpha_0) = (c_4 + c_5) \vee \alpha_0$. As noted after (2.1), this implies relation (2.1) for any $\alpha > \alpha_0$. Hence $(\mathbf{D}, F)$ is a reference form of $(E^{(\tau)}, F)$.

Since $(\mathbf{D}, H^1(\overline{D})$ is closed, the closedness of $(E^{(\tau)}, F)$ is clear. Furthermore, for any $\varphi \in F$ and positive constant a, $\varphi \wedge a \in F$,

$$\begin{aligned} E^{(\tau)}(\varphi \wedge a, \varphi \wedge a) &= \int_{D \cap \{\varphi \leq a\}} \sum_{i,j=1}^{d} a_{ij}(\tau, x) \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} dx \\ &\quad + \int_{D \cap \{\varphi \leq a\}} \sum_{i=1}^{d} b_i(\tau, x) \frac{\partial \varphi}{\partial x_i} \varphi(x) dx \\ &\quad + \int_{\partial D \cap \{\varphi \leq a\}} \sum_{i=1}^{d} \beta_i(\tau, x) \frac{\partial \varphi}{\partial x_i} \varphi(x) S(dx) \\ &= E^{(\tau)}(\varphi \wedge a, \varphi). \end{aligned}$$

Therefore, $(E^{(\tau)}, F$ satisfies the Markov property. The independence of α_0 and K of τ is clear from the definition. □

Since $\mathcal{C}^1(\overline{D})$ is dense in F, $(E^{(\tau)}, F)$ is a regular semi-Dirichlet form.

Lemma 2.4. *A compact subset B of $\overline{D}$ is zero capacity relative to $E^{(\tau)}$ for some and hence for all τ if and only if it is zero capacity relative to $\mathbf{D}$.*

Proof. Suppose that $\mathrm{Cap}^{(\tau),\alpha}(B) = 0$ for the α-capacity $\mathrm{Cap}^{(\tau),\alpha}$ of $E^{(\tau)}$. Then there exists a decreasing sequence $\{B_n\}$ of relatively compact open sets such that $B_n \supset B$ and $\lim_{n\to\infty} \mathrm{Cap}^{(\tau),\alpha}(B_n) = 0$. Let $e^{(\tau),\alpha}_{B_n}$ be a quasi-continuous modification of the α-equilibrium potential of B_n relative to $E^{(\tau)}$. Then for any function $f \in F$ satisfying $f \geq 1$ on a neighbourhood of B,

$$0 = \lim_{n\to\infty} \mathrm{Cap}^{(\tau),\alpha}(B_n) \geq K_\alpha^2 \lim_{n\to\infty} E^{(\tau)}_\alpha(e^{(\tau),\alpha}_{B_n}, e^{(\tau),\alpha}_{B_n})$$

for a constant K_α. By virtue of Lemma 2.2, this implies that $\lim_{n\to\infty} \mathbf{D}(e_{B_n}^{(\tau),\alpha}, e_{B_n}^{(\tau),\alpha}) = 0$ which yields that B is of zero capacity relative to $\mathbf{D}$.

Conversely, if B is of zero capacity relative to $\mathbf{D}$, then there exists a sequence $\{\varphi_n\} \subset F$ such that $\varphi_n \geq 1$ a.e. on a neighbourhood of B and $\lim_{n\to\infty} \mathbf{D}(\varphi_n, \varphi_n) = 0$. Since $E_\alpha^{(\tau)}(e_{B_n}^{(\tau),\alpha}, e_{B_n}^{(\tau),\alpha}) \leq K_\alpha^2 E_\alpha^{(\tau)}(\varphi_n, \varphi_n)$ for $\{B_n\}$ such that $B \subset B_n \subset \{x : \varphi_n U x) = 1\}$, Lemma2.2 implies that $\lim_{n\to\infty} \mathrm{Cap}^{(\tau),\alpha}(B_n) = 0$. □

For a positive q.c. function $h \in F$, put $A^{(\tau),h}(\varphi, \psi) = E^{(\tau)}(\varphi, \psi h)$. This can be defined if $\varphi, \psi \in F$ satisfies $\psi h \in F$. Define a symmetric Dirichlet form $C^{(\tau),h}$ defined on $F^h = \{\varphi \in L^2(\overline{D}; h \cdot m) : \nabla\varphi \in L^2(\overline{D}; h \cdot m)\}$ by

$$C^{(\tau),h}(\varphi, \psi) = \sum_{i,j=1}^{d} \int_D a_{ij}(\tau, x) \frac{\partial\varphi}{\partial x_i} \frac{\partial\psi}{\partial x_j} h(, x) dx.$$

If $\varphi \in F_b^h$, then $\varphi h \in L^2(D; m)$ and $\nabla(\varphi h) \in L^2(D; m)$. Hence $\varphi h \in F$ and $A^{(\delta),h}(\varphi, \psi) = E^{(\tau)}(\varphi, \psi h)$ is well defined and written as

$$\begin{aligned} A^{(\tau),h}(\varphi, \psi) &= C^{(\tau),h}(\varphi, \psi) + \sum_{i,j=1}^{d} \int_D a_{ij}(\tau, x) \frac{\partial\varphi}{\partial x_i} \frac{\partial h}{\partial x_j} \psi(x) dx \\ &\quad + \sum_{i=1}^{d} \int_D b_i(\tau, x) \frac{\partial\varphi}{\partial x_i} \psi(x) h(x) dx \\ &\quad + \int_{\partial D} \sum_{i=1}^{d} \beta_i(\tau, x) \frac{\partial\varphi}{\partial x_i} \psi(x) h(x) dS(x). \end{aligned} \tag{2.9}$$

Note that $\mathcal{C}^1(\overline{D}) \subset F_b \cap F^h$. Furthermore, for any $\varphi, \psi \in F_b \cap F^h$, $A^{(\tau),h}(\varphi, \psi)$ satisfies the following condition which implies the condition $(\mathcal{E}.6)$ given in Section 1.4 in [11].

Lemma 2.5. *For any $\alpha > \alpha_0$, there exist constants K_1 and K_2 satisfying*

$$\begin{aligned} |A^{(\tau),h}(\varphi, \psi)| &\leq K_1 \left(\|\psi\|_\infty E_\alpha^{(\tau)}(h, h)^{1/2} + E_\alpha^{(\tau)}(\psi, \psi)^{1/2} + C_\alpha^{(\tau),h}(\psi, \psi)^{1/2} \right) \\ &\quad \times \left(E_\alpha^{(\tau)}(\varphi, \varphi)^{1/2} + A_\alpha^{(\tau),h}(\varphi, \varphi)^{1/2} \right. \\ &\quad \left. + E_\alpha^{(\tau)}(\varphi, \varphi)^{1/4} E_\alpha^{(\tau)}(h, h)^{1/4} \right) \end{aligned} \tag{2.10}$$

for all $\varphi, \psi \in F_b \cap F^h$.

Proof. As the proof of equation (2.5),

$$\begin{aligned}\left|E^{(\tau),1}(\varphi,\psi h)\right| &\le C^{(\tau),h}(\varphi,\varphi)^{1/2}C^{(\tau),h}(\psi,\psi)^{1/2}\\&\quad+\lambda_2\|\psi\|_\infty\mathbf{D}(\varphi,\varphi)^{1/2}\mathbf{D}(h,h)^{1/2}\\&\quad+\|b\|_\infty C^{(\tau),h}(\varphi,\varphi)^{1/2}\|\psi\|_{h\cdot m}.\end{aligned}$$

Similarly, decompose $E^{(\tau),2}(\varphi,\psi h)$ as (2.7) by taking ψh instead of ψ. Denote the corresponding terms I, II and III by I',II' and III', respectively. Then, similarly to the proof of Lemma 2.2,

$$|\mathrm{II}'|\le\frac{1}{k_1}C^{(\tau),h}(\varphi,\varphi)^{1/2}\left(C^{(\tau),h}(\psi,\psi)^{1/2}+\frac{\|\psi\|_\infty}{\sqrt{c}}\mathbf{D}(h,h)^{1/2}\right)$$

and

$$|\mathrm{III}'|\le\frac{k_3}{k_1}\|\beta\|_\infty C^{(\tau),h}(\varphi,\varphi)^{1/2}\|\psi\|_{h\cdot m}.$$

Furthermore,

$$\begin{aligned}|\mathrm{I}'| &\le \frac{\|\overline{\beta}\|_\infty}{k_1}C^{(\tau),h}(\varphi,\varphi)^{1/2}C^{(\tau),h}(\psi,\psi)^{1/2}\\&\quad+\frac{\|\overline{\beta}\|_\infty\|\psi\|_\infty}{k_1}\mathbf{D}(\varphi,\varphi)^{1/2}\mathbf{D}(h,h)^{1/2}+\frac{k_3}{k_1}C^{(\tau),h}(\varphi,\varphi)^{1/2}\|\psi\|_{h\cdot m}.\end{aligned}$$

Combining these inequalities and noting that $\mathbf{D}$ is a reference form of $E^{(\tau)}$, we obtain that

$$\begin{aligned}|A^{(\tau),h}(\varphi,\psi)| &\le K_1\left(\|\psi\|_\infty E^{(\tau)}_\alpha(h,h)^{1/2}+E^{(\tau)}_\alpha(\psi,\psi)^{1/2}+C^{(\tau),h}_\alpha(\psi,\psi)^{1/2}\right)\\&\quad\times\left(E^{(\tau)}_\alpha(\varphi,\varphi)^{1/2}+C^{(\tau),h}_\alpha(\varphi,\varphi)^{1/2}\right)\end{aligned}\tag{2.11}$$

Similarly to the above estimate and the relation $ab\le(1/2)(a^2+b^2)$, there exists a constant k_5 satisfying

$$\begin{aligned}
A^{(\tau),h}(\varphi,\varphi) &= C^{(\tau),h}(\varphi,\varphi)+\frac{1}{2}\sum_{i,j=1}^{d}\int_D a_{ij}(\tau,x)\frac{\partial\varphi^2}{\partial x_i}\frac{\partial h}{\partial x_j}\,dx \\
&\quad +\frac{1}{2}\sum_{i=1}^{d}\int_D b_i(\tau,x)\frac{\partial\varphi^2}{\partial x_i}\,h(x)dx \\
&\quad +\frac{1}{2}\int_{\partial D}\sum_{i=1}^{d}\beta_i(\tau,x)\frac{\partial\varphi^2}{\partial x_i}h(x)dS(x) \\
&\geq C^{(\tau),h}(\varphi,\varphi)-k_5\Big(\|\varphi\|_\infty E_\alpha^{(\tau)}(\varphi,\varphi)^{1/2}E_\alpha^{(\tau)}(h,h)^{1/2} \\
&\qquad\qquad +C^{(\tau),h}(\varphi,\varphi)^{1/2}\|\varphi\|_{h\cdot m}\Big) \\
&\geq \frac{1}{2}C^{(\tau),h}(\varphi,\varphi)-k_5\|\varphi\|_\infty E_\alpha^{(\tau)}(\varphi,\varphi)^{1/2}E_\alpha^{(\tau)}(h,h)^{1/2} \\
&\qquad\qquad -\frac{1}{2}k_5^2(\varphi,\varphi)_{h\cdot m}.
\end{aligned}$$

Then, by (2.11) and the inequality $(a+b)^{1/2}\leq\sqrt{2}(\sqrt{a}+\sqrt{b})$, we obtain Equation (2.10). □

3 Associated space-time process and its application

By virtue of Theorem 2.3, there corresponds a time dependent semi-Dirichlet form$(\mathcal{E},\mathcal{F})$ on $\mathcal{H}$ defined by (1.7). Let $\mathbb{M}=(Z_t,\mathbb{P}_z)$ be the associated space-time Hunt process on Z, that is its resolvent $R_\alpha u$ is a q.c. function of $\mathcal{W}$ satisfying

$$\mathcal{E}_\alpha(R_\alpha u,v)=(u,v) \tag{3.1}$$

for all $\alpha>\alpha_0$ and $u\in\mathcal{H}$. Furthermore, there also exists a ν-dual resolvent $\widehat{R}_\alpha u\in\mathcal{W}$ satisfying $\mathcal{E}_\alpha(v,\widehat{R}_\alpha u)=(u,v)$ for $\alpha>\alpha_0$. By the resolvent equation, for any $\alpha>0$, $R_\alpha u$ and $\widehat{R}_\alpha u$ can be extended to $u\in L^\infty(Z;\nu)$ and $u\in L^1(Z;\nu)$ respectively.

Let $\mathcal{L}$ be the the generator corresponding to the resolvent $R_\alpha f$, that is $\mathcal{L}$ is defined for any function $u\in\mathcal{D}(\mathcal{L})=\{R_\alpha f: f\in\mathcal{F}',\alpha>\alpha_0\}$ by $\mathcal{L}u=\alpha u-f$ for $u=R_\alpha f\in\mathcal{D}(\mathcal{L})$. Since $\tau(t)=\tau(0)+t$, the transition function $p_t f\in\mathcal{W}$ can be expressed as $p_t f(\tau,x)=\int_{\overline{D}}p(\tau,x;\tau+t,dy)f(\tau+t,y)$ for a time-inhomogeneous transition function $p(\tau,x;t,dy)$. For any function $\varphi\in L^2(\overline{D};m)$, put $p(s,x;t,\varphi)=\int_{\overline{D}}p(s,x;t,dy)\varphi(y)$.

Lemma 3.1. *For any $\varphi \in L^2(\overline{D}; m)$ and $v \in \mathcal{F}$,*

$$-\left(\frac{\partial}{\partial \tau} p(\tau, x; t, \varphi), v(\tau, \cdot)\right) + E^{(\tau)}\left(p(\tau, \cdot; t, \varphi), v(\tau, \cdot)\right) = 0 \quad \text{a.e.}\ \tau. \tag{3.2}$$

Proof. Assume that $\varphi, v(\tau, \cdot) \in \mathcal{C}^1(\overline{D})$. For any $\xi(\tau) \in \mathcal{C}_0^1(\mathbb{R}^1)$, put $f(\tau, x) = \xi(\tau)\varphi(x)$. Then

$$\begin{aligned} \alpha R_\alpha f(\tau, x) &= \alpha \int_0^\infty \int_{\mathbb{R}^1} e^{-\alpha t} p(\tau, x; \tau + t, \varphi)\xi(\tau + t) d\tau dt \\ &= \int_0^\infty \int_{\mathbb{R}^1} e^{-\alpha t} \frac{\partial}{\partial t}\left(\xi(\tau + t) p(\tau, x; \tau + t, \varphi)\right) d\tau dt + \xi(\tau)\varphi(x). \end{aligned}$$

Hence, for $v(\tau, x) = \eta(\tau)\psi(x)$ with $\eta \in \mathcal{C}_0^1(\mathbb{R}^1)$ and $\psi \in \mathcal{C}^1(\overline{D})$,

$$\begin{aligned} 0 &= \mathcal{E}_\alpha(R_\alpha f, v) - (f, v) \\ &= -\int_0^\infty \int_{\mathbb{R}^1} e^{-\alpha t} \frac{\partial}{\partial \tau}\left(\xi(\tau + t) p(\tau, \cdot; \tau + t, \varphi), \psi\right) \eta(\tau) d\tau dt \\ &\quad + \int_0^\infty \int_{\mathbb{R}^1} e^{-\alpha t} \xi(\tau + t) E^{(\tau)}\left(p(\tau, \cdot; \tau + t, \varphi), \psi\right) \eta(\tau) d\tau dt \\ &\quad + \int_0^\infty \int_{\mathbb{R}^1} e^{-\alpha t} \left(\frac{\partial}{\partial t}\left(\xi(\tau + t) p(\tau, \cdot; \tau + t, \varphi)\right), \psi\right) \eta(\tau) d\tau dt. \end{aligned}$$

This implies that

$$\begin{aligned} &-\frac{\partial}{\partial \tau}\left(\xi(\tau + t) p(\tau, \cdot; \tau + t, \varphi), \psi\right) + \xi(\tau + t) E^{(\tau)}\left(p(\tau, \cdot; \tau + t, \varphi), \psi\right) \\ &+ \left(\frac{\partial}{\partial t}\left(\xi(\tau + t) p(\tau, \cdot; \tau + t, \varphi)\right), \psi\right) = 0. \end{aligned}$$

Therefore

$$-\left.\left(\frac{\partial}{\partial \sigma} p(\sigma, \cdot; \tau + t, \varphi), \psi\right)\right|_{\sigma = \tau} + E^{(\tau)}\left(p(\tau, \cdot; \tau + t, \varphi), \psi\right) = 0$$

which yields (3.2). □

Lemma 3.2. *The measure $d\eta(\tau, x) = d\tau \otimes dS(x)$ is a smooth measure.*

Proof. For any $O = O^{(k)}$ and $f \in \mathcal{C}_0^1(O)$, similarly to the proof of Lemma 2.2,

$$\begin{aligned}\left|\int_{O\cap\partial D} f(x)dS(x)\right| &= \left|\int_{V\cap\{\xi:\xi_d=0\}} \overline{f}(\xi',0)\overline{S}(\xi',0)d\xi'\right| \\ &= \left|-\int_{\mathbb{R}^d} \frac{\partial}{\partial\xi_d}\left(\overline{f}(\xi)\overline{S}(\xi)\right)d\xi\right| \\ &\leq k_1\mathbf{D}(f,f)^{1/2} + k_3\|f\| \\ &\leq k_4\mathbf{D}_1(f,f)^{1/2}, \end{aligned} \tag{3.3}$$

for $k_4 = k_1 \vee k_3$. This implies that dS is a smooth measure relative to $\mathbf{D}$ and hence relative to $E^{(\tau)}$ for all τ.

Let B be a compact subset of $O\cap\partial D$ and $\{B_n\}$ be a decreasing sequence of relatively compact open sets of such that $B = \cap_n B_n$. Then for any functions $\xi(\tau) \in \mathcal{C}_0^1(\mathbb{R}^1)$ and $f \in \mathcal{C}^1(O\cap\overline{D})$ satisfying $\xi \geq 1$ on an interval (a,b) and $f \geq 1$ on a neighbourhood of B, the α-capacity $\mathrm{Cap}^{(\alpha)}(\Lambda_n)$ of $\Lambda_n = (a,b)\times B_n$ relative to $(\mathcal{E},\mathcal{F})$ satisfies

$$\begin{aligned}\mathrm{Cap}^{(\alpha)}(\Lambda_n) &= \mathcal{E}_\alpha(\xi\otimes f, \widehat{e}^{\alpha}_{\Lambda_n}) \\ &= -\int_{\mathbb{R}^1}\xi'(\tau)\left(f,\widehat{e}^{\alpha}_{\Lambda_n}(\tau,\cdot)\right)d\tau + \int_{\mathbb{R}^1}E^{(\tau)}\left(f,\widehat{e}^{\alpha}_{\Lambda}(\tau,\cdot)\right)\xi(\tau)d\tau.\end{aligned}$$

If $(a,b)\times B$ is of zero capacity, then there exists a sequence $\{B_n\}$ satisfying $\lim_{n\to\infty}\mathrm{Cap}^{(\alpha)}(\Lambda_n) = 0$. Then, by choosing a subsequence if necessary, $\lim_{n\to\infty}\left(f,\widehat{e}^{\alpha}_{\Lambda_n}(\tau,\cdot)\right) = 0$ and $\lim_{n\to\infty}E^{(\tau)}\left(f,\widehat{e}^{\alpha}_{\Lambda}(\tau,\cdot)\right) = 0$ a.e.τ. Therefore, B is of zero capacity relative to $E^{(\tau)}$ and hence $\eta((a,b)\times B) = 0$ by Lemma 2.4.

By virtue of (3.3), for any $u(\tau,x) \in \mathcal{C}^1((a,b)\times O)$,

$$\left|\int_{\mathbb{R}^1}\int_{\partial D}u(\tau,x)dS(x)d\tau\right| \leq k_4\int_{\mathbb{R}^1}E^{(\tau)}\left(u(\tau,\cdot),u(\tau,\cdot)\right)^{1/2}d\tau \leq k_4\|u\|_{\mathcal{W}}.$$

This implies that $d\tau\otimes dS$ is a smooth measure relative to $(\mathcal{E},\mathcal{F})$ □

Lemma 3.2 implies that there exists a positive continuous additive functional ℓ_t satisfying

$$\lim_{t\to 0}\mathbb{E}_{\widehat{h}\cdot\nu}\left(\int_0^t u(Z_t)d\ell_t\right) = \int_{\mathbb{R}^1\times\partial D}\widehat{h}(\tau,x)u((\tau,x)d\tau dS(x) \tag{3.4}$$

for any q.c function u and α-coexcessive function $\widehat{h}$. The functional ℓ_t is called the local time of ∂D.

For any fixed $T < \infty$ and $f(\tau, x) \in L^2([0,T] \times \overline{D}; \nu)$,

$$R^T f(\tau, x) = \mathbb{E}_{(\tau,x)} \left(\int_\tau^T f(t, X_t) dt \right)$$

is the 0-order potential of $1_{[0,T]}(\tau) f(\tau, x)$. Hence it satisfies $\mathcal{E}(R^T f, v) = (f, v)$ for any $v \in \mathcal{F}$. Since v is arbitrary, this implies

$$\begin{aligned} &- \left(\frac{\partial}{\partial \tau} R^T f(\tau, \cdot), v(\tau, \cdot) \right) + E^{(\tau)}(R^T f(\tau, \cdot), v(\tau, \cdot)) \\ &= \left(1_{[0,T]} f(\tau, \cdot), v(\tau, \cdot) \right) \end{aligned} \tag{3.5}$$

for a.e. $\tau < T$. Similarly, for $g(\tau, x) \in L^2([0,T] \times \partial D; \eta)$ with $d\eta(\tau, x) = d\tau dS(x)$, the potential of the measure $U^T(g \cdot \eta)$ given by

$$U^T(g \cdot \eta)(\tau, x) = \mathbb{E}_{(\tau,x)} \left(\int_\tau^T g(Z_t) d\ell_t \right)$$

satisfies, for any $v \in \mathcal{W}$,

$$\begin{aligned} &\left(U^T(g \cdot \eta)(\tau, \cdot), \frac{\partial}{\partial \tau} v(\tau, \cdot) \right) + E^{(\tau)}(U^T(g \cdot \eta)(\tau, \cdot), v(\tau, \cdot)) \\ &= \langle g(\tau, \cdot) \cdot S, v(\tau, \cdot) \rangle. \end{aligned} \tag{3.6}$$

Theorem 3.3. *Let $f \in L^2([0,T] \times \overline{D}; \nu), g \in L^2([0,T] \times \partial D; \eta)$ and $\varphi \in L^2(\overline{D}; m)$. Then, the function $u(\tau, x)$ defined by*

$$\begin{aligned} u(\tau, x) \quad = \quad & \mathbb{E}_{(\tau,x)} (\varphi(X_T)) + \mathbb{E}_{(\tau,x)} \left(\int_\tau^T f(t, X_t) dt \right) \\ & + \mathbb{E}_{(\tau,x)} \left(\int_\tau^T g(t, X_t) d\ell_t \right) \end{aligned} \tag{3.7}$$

satisfies

$$\begin{aligned} &\int_0^T \left(u(\tau, \cdot), \frac{\partial v}{\partial \tau} \right) d\tau + \int_0^T E^{(\tau)} \left(u(\tau, \cdot), v(\tau, \cdot) \right) \\ &= \int_0^T \int_D f(\tau, x) v(\tau, x) d\tau dx + \int_0^T \int_{\partial D} g(\tau, x) v(\tau, x) d\tau dS(x) \end{aligned} \tag{3.8}$$

for any $v \in \mathcal{C}^1((0,T) \times \overline{D})$ such that $v(\tau, x) = 0$ for any τ outside of a compact subset of $(0,T)$.

Proof. Since $u(\tau,x) = p(\tau,x;T,\varphi)+R^T f(\tau,x)+U^T(g\cdot\eta)(\tau,x)$, (3.8) follows from (3.2), (3.5) and (3.6). $\square$

Corollary 3.4. *The function $u(\tau,x)$ given by (3.7) satisfies the following equations in the distribution sense.*

$$\frac{\partial u}{\partial \tau} + \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}(\tau,x)\,\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{d} b_i(\tau,x)\,\frac{\partial u}{\partial x_i} = -f(\tau,x) \tag{3.9}$$

for $(\tau,x) \in (0,T)\times D$,

$$\sum_{i=1}^{d}\left(\sum_{j=1}^{d} a_{ij}(\tau,x)n_j(\tau,x) + \beta_i(\tau,x)\right)\frac{\partial u}{\partial x_i} = g(\tau,x) \tag{3.10}$$

for $(\tau,x) \in (0,T)\times\partial D$ and $u(T,x) = \varphi(x)$ for $x \in \overline{D}$, where $\mathbf{n} = (n_1,n_2,\cdots,n_d)$ is the inward unit normal vector of ∂D.

Proof. Put $v(\tau,x) = \xi(\tau)\psi(x)$ in (3.7). Then the lefthand side of (3.8) can be written by Green's theorem as

$$\begin{aligned}
&\int_0^T (u(\tau,\cdot),\psi)\,\xi'(\tau)d\tau + \int_0^T E^{(\tau)}\,(u(\tau,\cdot),\psi)\,\xi(\tau)d\tau \\
= \ & -\int_0^T\int_D \frac{\partial u}{\partial \tau}\psi(x)\xi(\tau)d\tau dx - \sum_{i=1}^{d}\int_0^T\int_D b_i(\tau,x)\frac{\partial u}{\partial x_i}\psi(x)\xi(\tau)d\tau dx \\
&-\sum_{i,j=1}^{d}\int_0^T\int_D \frac{\partial}{\partial x_j}\left(a_{ij}(\tau,x)\frac{\partial u}{\partial x_i}\right)\psi(x)\xi(\tau)d\tau dx \\
&+\sum_{i=1}^{d}\int_0^T\int_{\partial D}\left(\sum_{j=1}^{d} a_{ij}n_j - \beta_i\right)(\tau,x)\,\frac{\partial u}{\partial x_i}\,\psi(x)\xi(\tau)d\tau dS(x).
\end{aligned}$$

This compared with the righthand side of (3.8) gives the result. $\square$

For $\delta > \alpha_0$, let us fix a δ-coexcessive function $\widehat{h}_\delta \in \mathcal{F}$ which is bounded from below by a positive constant on any compact set of Z. See [11] Theorem 2.4.8 for the existence of such function. In [11], concerning to the stochastic calculus related to time dependent Dirichlet forms, only Fukushima's decomposition in the weak sense and its direct consequences are mentioned. But, most of the calculus similar to the time independent case as in Chapter 5 in [11] are possible. In particular, under our present settings, strong sense

Fukushima's decomposition as Theorem 5.1.4 is possible. Let us briefly mention about the outline of it. The energy $e^{(\delta)}(A)$ of an additive functional A_t is defined by

$$e^{(\delta)}(A) = \lim_{\beta\to\infty} \beta^2 \mathbb{E}_{\widehat{\nu}} \left(\int_0^\infty e^{-\beta t} A_t^2 dt \right), \tag{3.11}$$

where $\widehat{\nu} = \widehat{h}_\delta \cdot \nu$. For a q.c. function $u \in \mathcal{W}$, put $A_t^{[u]} = u(Z_t) - u(Z_0)$. If $u \in \mathcal{W}_b$, then $u\widehat{h}, u^2 \in \mathcal{F}$ and the energy of $A^{[u]}$ is given by with energy

$$\begin{aligned} e^{(\delta)}(A^{[u]}) &= 2 \lim_{\beta\to\infty} \beta \left(u - \beta R_\beta u, u\widehat{h}_\delta \right) - \lim_{\beta\to\infty} \beta \left(u^2 - \beta R_\beta u^2, \widehat{h}_\delta \right) \\ &= 2\mathcal{E}(u, u\widehat{h}_\delta) - \mathcal{E}(u^2, \widehat{h}_\delta) = 2\mathcal{B}(u, u\widehat{h}_\delta) - \mathcal{B}(u^2, \widehat{h}_\delta). \end{aligned} \tag{3.12}$$

In the preveous section, for a strictly positive function $h \in F$, we introduced a Dirichlet form $(C^{(\tau),h}, F^h)$. By taking $\widehat{h}_\delta(\tau, \cdot)$ instead of h for each fixed τ, define a symmetric Dirichlet form $(C^{(\tau),\delta}, F^{(\tau),\delta})$ similarly by $F^{(\tau),\delta} = \{\varphi \in L^2(D : \widehat{h}_\delta(\tau, \cdot) \cdot m, \nabla\varphi \in L^2(D; \widehat{h}_\delta(\tau, \cdot) \cdot m)$ and

$$C^{(\tau),\delta}(\varphi, \psi) = \int_D \sum_{i,j=1}^d a_{ij}(\tau, x) \frac{\partial\varphi}{\partial x_i} \frac{\partial\psi}{\partial x_j} \widehat{h}_\delta(\tau, x)\, dx.$$

Let

$$\mathcal{C}^\delta(u, v) = \int_{\mathbb{R}^1} C^{(\tau),\delta} \left(u(\tau, \cdot), v(\tau, \cdot) \right) d\tau$$

and

$$\mathcal{A}^\delta(u, v) = \mathcal{B}(u, u\widehat{h}_\delta) = \int_{\mathbb{R}^1} A^{(\tau),\delta} \left(u(\tau, \cdot), v(\tau, \cdot) \right) d\tau$$

Let $\overset{\circ}{\mathcal{M}}$ and $\mathcal{N}$ be the families of martingale additive functionals of finite energy and continuous additive functional of zero energy relative to $\widehat{h}_\delta$. Then the following Fukushima's decomposition of $A_t^{[u]} = \widetilde{u}(Z_t) - \widetilde{u}(Z_0)$ holds.

Theorem 3.5. *For any $u \in \mathcal{W}_b$ such that $u\widehat{h}_\delta \in \mathcal{F}$, there exist uniquely $M^{[u]} \in \overset{\circ}{\mathcal{M}}$ and $N^{[u]} \in \mathcal{N}$ such that*

$$A_t^{[u]} = M_t^{[u]} + N_t^{[u]}. \tag{3.13}$$

Proof. Suppose that $u \in \mathcal{W}_b$ satisfies $u\widehat{h}_\delta \in \mathcal{F}$. Then $u(\tau, \cdot) \in F^{(\tau),\delta}$ for all τ. Hence, Lemma 2.5 yields that

$$\begin{aligned} \mathcal{A}^\delta(u, u)^2 &\le K_2 \left(\|u\|_\infty^2 \mathcal{B}_\alpha(\widehat{h}_\delta, \widehat{h}_\delta) + \mathcal{B}_\alpha(u, u) + \mathcal{C}_\alpha^\delta(u, u) \right) \\ &\quad \times \left(\mathcal{B}_\alpha(u, u) + \mathcal{A}_\alpha^\delta(u, u) + \mathcal{B}_\alpha^\delta(u, u)^{1/2} \mathcal{B}_\alpha^\delta(\widehat{h}_\delta, \widehat{h}_\delta)^{1/2} \right) \end{aligned} \tag{3.14}$$

where $\mathcal{C}^\delta_\alpha(u,v) = \mathcal{C}^\delta(u,v) + \alpha(u,v)_{\widehat{\nu}}$. It also holds that

$$\left|\mathcal{B}(u^2, \widehat{h}_\delta)\right| \le K_2 \|u\|_\infty \mathcal{B}_\alpha(u,u)^{1/2} \mathcal{B}_\alpha(\widehat{h}_\delta, \widehat{h}_\delta)^{1/2}$$

for some constant K_2 depending on α. Therefore, if a sequence $\{u_n\}$ converges to u relative to $\mathcal{B}_\alpha + \mathcal{A}^\delta_\alpha$, then it also converges relative to the energy $e^{(\delta)}$ by (3.12).

For any $u \in \mathcal{W}$, since

$$\begin{aligned}
\mathcal{B}\left(\alpha R_\beta u, \beta R_\beta u\right) &= \mathcal{E}\left(\beta R_\beta u, \beta R_\beta u\right) = \beta\left(u - \beta R_\beta u, \beta R_\beta u\right) \\
&= \beta\left(u - \beta R_\beta u, u\right) - \beta\left(u - \beta R_\beta u, u - \beta R_\beta u\right) \\
&\le \beta\left(u - \beta R_\beta u, u\right) = \mathcal{E}\left(\beta R_\beta u, u\right) \\
&\le K \mathcal{E}_{\beta_0}(\beta R_\beta u, \beta R_\beta u)^{1/2} \mathcal{E}_{\beta_0}(u,u)^{1/2} \\
&\le K \mathcal{B}_{\beta_0}(\beta R_\beta u, \beta R_\beta u)^{1/2} \mathcal{B}_{\beta_0}(u,u)^{1/2}
\end{aligned}$$

we can show as Theorem 1.1.4 of [11], that that $\lim_{\beta\to\infty} \beta R_\beta u = u$ relative to $\mathcal{B}_\alpha$ for $\alpha > \alpha_0$. In the above inequality, we used the sector condition of $\mathcal{B}$. Instead of sector condition, by using (3.14), we can also show that $\lim_{n\to\infty} nR_n u = u$ relative to $\mathcal{A}^\delta_\alpha$. Therefore, for $u_n = nR_n u$ with $u \in \mathcal{W}_b$, (3.12) implies that $\lim_{n\to\infty} e^{(\delta)}(A^{[u_n]} - A^{[u]}) = 0$. Hence, similarly to Theorem 5.1.4 in [11], we can obtain the Fukushima's decomposition. □

Since any function $w \in \mathcal{C}^1_0(\mathbb{R}^1 \times \overline{D})$ satisfies the condition of Theorem 3.5, the decomposition $A^{[w]}_t = M^{[w]}_t + N^{[w]}_t$ holds. The martingale part $M^{[w]}$ and the zero energy part $N^{[w]}$ are characterized as follows.

Corollary 3.6. *For any $w \in \mathcal{C}^1_0(\mathbb{R}^1 \times \overline{D})$, the martingale part $M^{[w]}$ satisfies*

$$\langle M^{[w]} \rangle_t = \int_0^t \sum_{i,j=1}^d a_{ij}(Z_s) \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j}(Z_s)\, ds. \tag{3.15}$$

Also the zero energy part $N^{[w]}$ satisfies,

$$\begin{aligned}
\lim_{t\to 0} \frac{1}{t} \mathbb{E}_{v\cdot\nu}\left(N^{[w]}_t\right) &= \left(\frac{\partial w}{\partial \tau}, v\right) - \int_Z \sum_{i,j=1}^d a_{ij}(\tau,x) \frac{\partial w}{\partial x_i} \frac{\partial v}{\partial x_j} \nu(\tau,x) \\
&+ \int_Z \sum_{i=1}^d b_i(\tau,x) \frac{\partial w}{\partial x_i} v(\tau,x) d\nu(\tau,x) \\
&+ \int\int_{\mathbb{R}^1\times\partial D} \sum_{i=1}^d \beta_i(\tau,x) \frac{\partial w}{\partial x_i} v(\tau,x) d\tau dS(x)
\end{aligned} \tag{3.16}$$

for any $v \in \mathcal{F}_b$ with compact support.

Proof. Since v is bounded by a constant multiple of $\widehat{h}_\delta$, $\lim_{t\to 0}(1/t)\mathbb{E}_{v\cdot\nu}(N^{[w]}) = 0$. Hence

$$\begin{aligned}
\int_Z v(z)d\mu_{\langle w\rangle}(z) &= \lim_{t\to 0}\frac{1}{t}\mathbb{E}_{v\cdot\nu}\left((w(Z_t)-w(Z_0))^2\right. \\
&= \lim_{t\to 0}\frac{1}{t}\left(2(w-p_tw,vw)_\nu-(w^2-p_tw^2,v)\right) \\
&= 2\mathcal{B}(w,wv)-\mathcal{B}(w^2,v) \\
&= 2\int_Z\sum_{i,j=1}^d a_{ij}(z)\frac{\partial w}{\partial x_i}\frac{\partial wv}{\partial x_j}\,d\nu(z) \\
&\quad -2\int_Z\sum_{i,j=1}^d a_{ij}(z)\frac{\partial w^2}{\partial x_i}\frac{\partial v}{\partial x_j}\,d\nu(z) \\
&= 2\int_Z\sum_{i,j=1}^d a_{ij}(z)\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}\,d\nu(z).
\end{aligned}$$

Therefore $\mu_{\langle w\rangle}(dz) = 2\sum_{i,j=1}^d a_{ij}(z)\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}\,d\nu(z)$ which implies (3.13). Similarly, (3.14) follows from

$$\lim_{t\to 0}(1/t)\mathbb{E}_{v\cdot\nu}(N^{[w]}) = \lim_{t\to 0}(1/t)\mathbb{E}_{v\cdot\nu}\left(w(Z_t)-w(Z_0)\right) = -\mathcal{E}(w,v).$$

□

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